Notes for Macroeconomics III

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Contents

General macroeconomic micro-foundations

Macroeconomic relationships are derived from first principles in this course.

Household preferences In the simplest setting, households optimize expected utility over consumption, c_t and leisure, x_t.

$$U = E_0 \left[\sum_{t=0}^{T} \beta^t u(c_t, x_t) \right]$$

The utility function, u, continuously differentiable, displays decreasing marginal utility, and c_t is essential.

The discount factor β represents the discount factor of future utility (the degree of patience). It is assumed to be larger than 0 and smaller than 1. $\beta = \frac{1}{1+\rho}$, where ρ is the discount rate.

Constant relative risk aversion utility function We usually assume CRRA-preferences, where σ is the degree of constant relative risk aversion, $\sigma = -c\frac{u''(c)}{u'(c)}$,

$$u(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & \sigma \geq 0, \quad \sigma \neq 1 \\ \ln(c) & \sigma = 1 \end{cases}$$

The intertemporal elasticity of substitution, which measures the strength of interest rate changes in consumption, is $\varepsilon = \frac{1}{\sigma}$. If the IES is large, consumption smoothing is preferred and the substitution effect dominates. Concavity of the utility function rises with σ. When we assume log-utility, the income and substitution effects of interest rate changes cancel each other out. The income effect dominates when $\sigma > 1$ and vice versa.

The production function We assume that $f(K, L)$ displays constant returns to scale. A common functional form is the Cobb-Douglas production function.

The dynamic budget constraint Assuming an inelastic labour supply and ruling out risk and leisure, the DBC is given as

$$a_{t+1} = a_t R_t + w_t - c_t \Leftrightarrow \Delta a_{t+1} = a_t r_t + w_t - c_t,$$

where $R_t = 1 + r_t - \delta$ is the gross interest rate, a_t is assets from which the household receives interest incomes, w_t is salary, and c_t is consumption.

The intertemporal budget constraint This signifies that all financially discounted income must equal all financially discounted consumption. For two periods, assuming $a_0 = a_2 = 0$, this is

$$c_0 + \frac{c_1}{R_1} = w_0 + \frac{w_1}{R_1}.$$

The price of one unit of second period consumption is $\frac{1}{R_t}$ units of first period consumption. Thus, the price of consumption in period 2 is decreasing in R_1.

The Euler equation This represents the first order condition for optimality. It can be derived through a Lagrangian or substitution. It follows that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1}.$$

The slope of the consumption path (the intertemporal willingness of substitution) is thus determined by three factors. If patience rises, $\beta \uparrow$, more value is placed on future consumption, $\frac{c_{t+1}}{c_t} \uparrow$. A higher interest rate, $R_{t+1} \uparrow$, renders future consumption cheaper, $\frac{c_{t+1}}{c_t} \uparrow$. Lastly, a larger curvature of the marginal utility function, $\sigma \uparrow$, yields a lesser degree of changes in the consumption path of changes in β and R_{t+1}.

The Euler equation reveals three effects of interest rate changes. The negative first income effect (also known as the wealth effect), as $R_{t+1} \uparrow$ lowers the discounted value of future income and thus total consumption, the positive second income effect, as $R_{t+1} \uparrow$ lowers the price of a given consumption bundle, and the substitution effect towards the cheaper good such that $R_{t+1} \uparrow$ means $\frac{c_{t+1}}{c_t} \uparrow$.

Ramsey model

The model The Ramsey model is a neoclassical growth model and explains long run growth. Aggregate dynamics are micro-founded, unlike the Solow model, and explains aggregate capital accumulation, savings schedule, output, and consumption. It assumes rational expectations.

Infinite horizon We consider the limiting case of optimization for $T \to \infty$ for the representative agent for three reasons. Inter-generational altruism, time-invariant survival probability, and mathematical simplicity. The representative agent assumption makes the aggregation of individuals trivial.

The representative firm Taking rental rates and wages as given, the representative firm maximizes

$$\max_{K_t, L_t} f(K_t, L_t) - K_t r_t - L_t w_t.$$

The model features perfect competition such that the factor prices are equal to the corresponding marginal products of the production function and profits, z_t, are 0. The firms budget constraint is $f(K_t, L_t) = K_t r_t + L_t w_t + z_t$. Market clearing satisfies that $L_t = 1$ and $K_t = k_t$.

No-Ponzi game It states that you cannot finance consumption through debt indefinitely,

$$\lim_{T\to\infty} q_T a_{T+1} \geq 0,$$

where

$$q_T = (R_0 \cdot R_1 \cdot \dots \cdot R_T)^{-1}$$

is the opportunity cost of consumption in utility terms at time T.

Transversality condition This limits the NPG condition in optimum such that

$$\lim_{T\to\infty} q_T a_{T+1} = 0 \Leftrightarrow \lim_{T\to\infty} \beta^T u'(c_T) k_{t+1} = 0,$$

which gives that activities at the end of time is 0.

Optimization problem and Euler equation Assuming a population growth of n, the representative agent satisfies

$$\max_{c_t} \sum_{t=0}^{T} \beta^t (1+n)^t u(c_t).$$

The corresponding dynamic budget constraint is

$$a_{t+1}(1+n) = a_t(1+r_t-\delta) + w_t - c_t + z_t - T_t$$

with $a_t = k_t + b_t$, where a_t are assets, $R_t = (1+r_t-\delta)$ is the gross rental rate, w_t is wage, c_t is consumption, $z_t = 0$ is firm profit, T_t is a lump sum tax, $(1+n)$ is population growth, and b_t is government debt. The government budget constraint is $b_{t+1}(1+n) = G_t - T_t + R_t b_t$.

The Lagrangian thus yields

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t (1+n)^t [u(c_t) + \lambda_t(a_t R_t + w_t - c_t - T_t - a_{t+1}(1+n))].$$

Taking the F.O.C.'s and combining gives

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow \beta^t(1+n)^t u'(c_t) = \lambda_t \beta^t (1+n)^t \Leftrightarrow u'(c_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Leftrightarrow \lambda_t (1+n)^{t+1} \beta^t = \beta^{t+1}(1+n)^{t+1} \lambda_{t+1} R_{t+1} \Leftrightarrow \lambda_{t+1}\beta R_{t+1} = \lambda_t$$
$$\Rightarrow \beta R_{t+1} u'(c_{t+1}) = u'(c_t) \Leftrightarrow \frac{u'(c_t)}{u'(c_{t+1})} = \beta R_{t+1},$$

which is the Euler equation.

Laws of motion The laws of motion for capital and consumption are determined from the dynamic budget constraint and the Euler equation.

The law of motion for capital is derived through the DBC as

$$(1+n)a_{t+1} = a_t(1+r_t-\delta) + w_t - c_t + z_t - T_t, \quad a_t = k_t + b_t, \quad b_{t+1}(1+n) = G_t - T_t + (1+r_t-\delta)b_t \Leftrightarrow$$
$$(1+n)k_{t+1} = k_t(1+r_t-\delta) + w_t - c_t + f(k_t,1) - k_t r_t - w_t - T_t + b_t(1+r_t-\delta) - (G_t - T_t + (1+r_t-\delta)b_t)$$
$$\Leftrightarrow (1+n)k_{t+1} = k_t(1-\delta) - c_t + f(k_t,1) - G_t.$$

If c_t is large, then a high level of consumption leaves little output left to invest. Thus, capital growth falls and vice versa.

The law of motion for consumption is derived through the Euler as

$$u'(c_t) = \beta(1 + f'(k_t,1) - \delta)u'(c_{t+1}).$$

A high level of capital yields a low marginal product and thus a low interest rate. Therefore, we would rather bring consumption forward such that future consumption growth falls and vice versa.

Steady state Imposing steady state such that $c_t = c_{t+1}$ and $k_t = k_{t+1}$ yields the following equations, defined the phase-diagram:

$$c_t = f(k_t,1) - G_t - k_t(n+\delta)$$
$$1 = \beta(1 + f'(k_t,1) - \delta)$$

Specifying a functional form for the production function such that $f'(k_t,1) = \frac{1}{\beta} - (1-\delta)$ solves the steady state.

Golden rule We note from the solution that $k^\star$ must be below the Golden Rule level from the Solow growth model:

$$f'(k^\star,1) = \delta + \frac{1}{\beta} - 1 > \delta = f'(k^{gr},1) \Rightarrow k^\star < k^{gr}$$

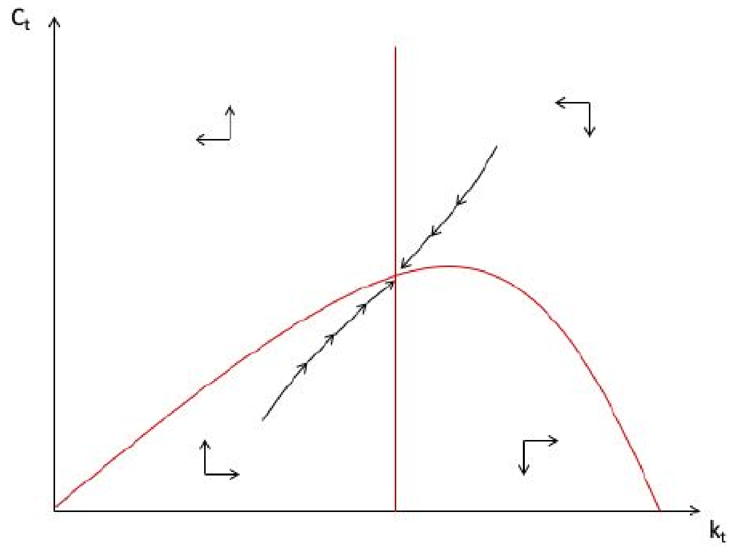
This is caused by the micro-foundations of the intertemporally utility optimizing agents. If $\beta \to 1$ then $k^\star \to k^{gr}$.

Welfare Equilibrium in the Ramsey model is Pareto efficient, assuming no externalities, due to the first welfare theorem[1]. It is never optimal to accumulate capital above the balanced growth path and dynamic efficiency holds in the Ramsey model.

Transitory growth There exists a balanced growth path at point E and a unique saddle path for any k_0

[1]If markets are competitive and complete and there are no externalities (and if the number of agents is finite), then the decentralized equilibrium is Pareto-efficient that is, it is impossible to make anyone better of without making someone else worse of.

that yields convergence towards steady state.

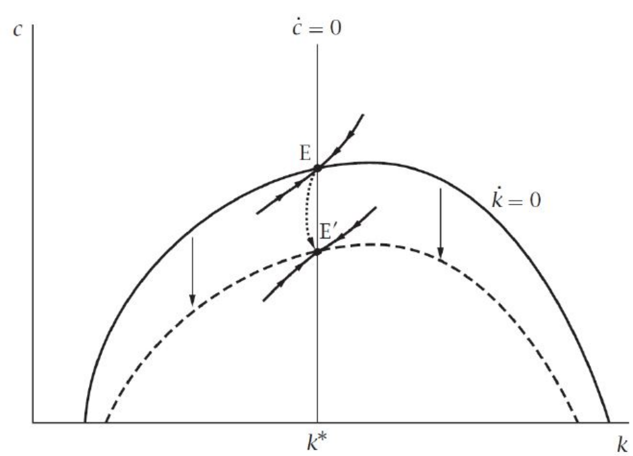


If we are not on the saddle path, we either end up on it, converge towards $k_t = 0$, or towards $c_t = 0$.

For $k_t = 0$: If consumption is too high relative to the saddle path, capital decreases. This increases consumption growth, increasing future consumption. This will depress capital accumulation further, and we end up in a situation, where agents consume all income.

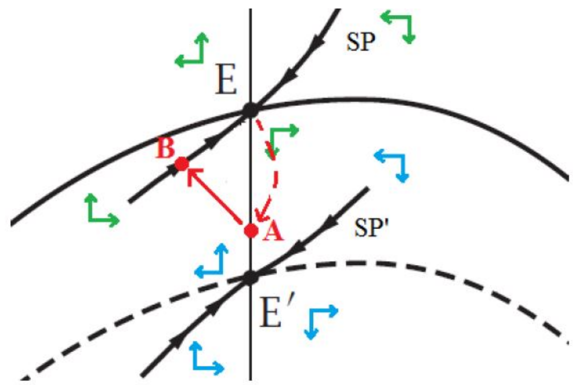
For $c_t = 0$: If consumption is too low, capital increases, since savings are high. This reduces interest rates, which reduces the value of future consumption, causing a further decline in consumption. This continues until $c_t = 0$, i.e. all income is invested in capital.

Introducing government spending If government spending (fiscal policy) is implemented at time t_0 and indefinitely forwards, the adjustment will happen instantaneously through consumption as capital cannot jump.



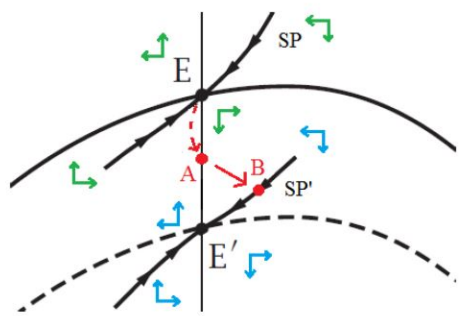
Aggregate output and capital accumulation is unaffected and we see public spending 1:1 crowding out private spending.

If the increase in government spending happens unexpectedly at time t_0 and ends at time t_1, the dynamics are as follows:



At time t_0, the jump happens to A and the blue arrows govern the dynamics. At time t_1 and forwards, the green arrows govern dynamics and we see convergence back towards E. Thus, we see a crowding out of private investment and a suppression of private consumption.

If the increase in government spending is announced at time t_0, happens expectedly at time t_1, and ends at time t_2, the dynamics are as follows:



A jump to A happens at t_0. Then, at t_1, the blue arrows govern dynamics towards E', from which, at t_2, the green arrows govern convergence back towards E. Households expect lower income in the future, so it is optimal to start adjusting consumption downwards now. This implies temporarily higher capital accumulation and output.

Ricardian equivalence It makes no difference to the success of stimulus and equilibrium allocation if it is founded with debt or taxes due to the Ricardian equivalence. Important assumptions for this to hold is

- Lump-sum taxation
- Infinite time horizon for households
- Closed economy

- No risk of default
- Unproductive government spending

If one or more of these assumptions do not hold, then Ricardian equivalence is not valid.

Distortionary taxation Distortionary taxation yields behavioral changes. Taxation on capital income is such that $T_t = \tau_t r_t a_t$. The equilibrium law of motion for capital is unaffected, as firms take the tax as given when optimizing but consumer behaviour is changed. The Euler equation shifts and is now

$$u'(c_t) = \beta(1 + \tilde{r}_t - \delta)u'(c_{t+1}), \tag{1}$$

where $\tilde{r}_t = (1 - \tau)r_t$. The after-tax return on capital must still be equal to $\delta + \frac{1}{\beta} - 1$ on the balanced growth path, which lowers $k^\star$ as the law of motion for consumption is shifted to the left.

Overlapping generations model

The model The OLG model is a long run growth model. It incorporates population turnover, and analyzes how life-cycle considerations affect capital accumulation, savings schedule, output, and consumption, as well as considering social security systems (pension schemes). The OLG model can display dynamic inefficiency. Governments have infinite horizons, while the individual has a horizon of two periods.

Population dynamics The population is assumed to grow at a pace such that $L_{t+1} = (1 + n)L_t$. People live for two periods. One as young, where they work and save, and one as old, where they do not work and dissave. This means that there are L_t young people and $L_{t-1} = \frac{L_t}{1+n}$ old people alive at time t.

Preferences Utility is derived from consumption while alive, so

$$U_t = u(c_{1t}) + \frac{1}{1+\rho}u(c_{2t+1}),$$

where $\frac{1}{1+\rho} = \beta$.

Budget constraint In the simple case, the budget constraint for each period in life is

$$c_{1t} + s_t = w_t$$
$$c_{2t+1} = (1 + r_{t+1} - \delta)s_t,$$

which combines to the lifetime budget constraint of

$$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1} - \delta} = w_t.$$

The representative firm It operates in the same fashion as in the Ramsey model such that

$$r_t = f'(k_t) \quad \text{and} \quad w_t = f(k_t) - f'(k_t)k_t,$$

where $k = \frac{K}{L}$.

Optimization problem Combining the utility function and the budget constraints yield

$$\max_{c_{1t}, c_{2t+1}} u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$
$$\text{subject to} \quad c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1} - \delta} = w_t$$
$$\text{or subject to} \quad c_{1t} + s_t = w_t \quad \text{and} \quad c_{2t+1} = (1 + r_{t+1} - \delta)s_t,$$

which can be solved either through substitution or through the Lagrangian.

Setting up the Lagrangian and taking F.O.C.'s gives

$$\mathcal{L} = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1}) + \lambda \left[w_t - c_{1t} - \frac{c_{2t+1}}{1 + r_{t+1} - \delta} \right]$$
$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0 \Leftrightarrow u'(c_{1t}) = \lambda \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0 \Leftrightarrow \frac{1}{1+\rho} u'(c_{2t+1}) = \lambda \frac{1}{1 + r_{t+1} - \delta}.$$

Combining these yields the Euler equation of

$$u'(c_{1t}) = \frac{1 + r_{t+1} - \delta}{1+\rho} u'(c_{2t+1})$$

The same result can be obtained by substituting an expression for s_t into the utility function and maximizing w.r.t. s_t:

$$\max_{s_t} u(w_t - s_t) + \frac{1}{1+\rho} u(s_t(1 + r_{t+1} - \delta)) \Rightarrow u'(w_t - s_t) = \frac{1 + r_{t+1} - \delta}{1+\rho} u'(s_t(1 + r_{t+1} - \delta))$$

Thus, we equate the marginal cost of giving up one unit of consumption today with the marginal benefit of consuming it tomorrow, while considering interest and discount. From the fraction, $\frac{1+r_{t+1}-\delta}{1+\rho}$, we see a positive substitution effect of changes in the interest rate to savings (save more due to higher returns), while we see a negative income effect from $s_t(1 + r_{t+1} - \delta)$ (save less due to higher income).

Assuming CRRA utility yields

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1 + r_{t+1} - \delta}{1+\rho} \right)^{\frac{1}{\sigma}}.$$

Consumption grows, when the interest rate is large relative to the discount rate. This effect is amplified, when σ is low, due to a high intertemporal elasticity of substitution and a dominating substitution effect.

Capital accumulation The capital accumulation schedule is determined by the optimal savings rate, defined by the solution w.r.t. savings, as

$$k_{t+1}(1+n) = s_t,$$

which implicitly defines a law of motion such that

$$k_{t+1}(1+n) = s(w_t, r_{t+1}) = s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}).$$

The log-CD case Assuming log-utility and a Cobb-Douglas production function we get

$$\frac{c_{2t+1}}{c_{1t}} = \frac{1 + r_{t+1} - \delta}{1+\rho}.$$

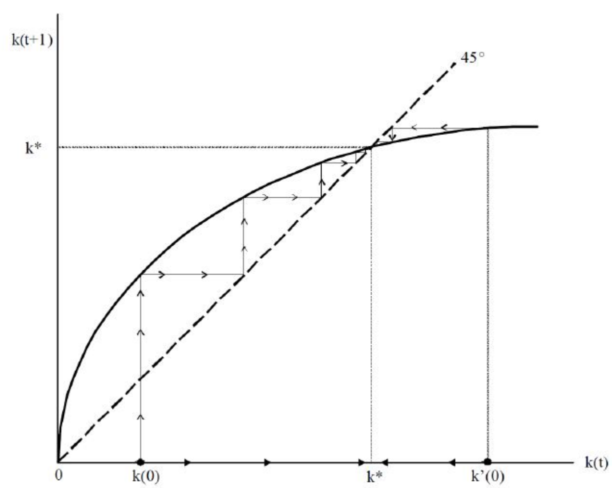
The corresponding savings schedule is a constant fraction of income, due to the substitution and income effects cancelling out;

$$s_t = \frac{1}{2+\rho} w_t,$$

which gives the capital accumulation schedule of

$$k_{t+1}(1+n) = s_t = \frac{1}{2+\rho} w_t = \frac{1}{2+\rho} k_t^{\alpha}(1-\alpha),$$

from which steady state can be derived.



Welfare Unlike the Ramsey model, households may very well accumulate capital to a level above the balanced growth path, when optimizing the golden rule. The GR level of capital maximizes

$$c = f(k) - nk,$$

being output minus break-event investment per person, which gives

$$\frac{\partial c}{\partial k} = 0 \Leftrightarrow f'(k^{gr}) = n.$$

In the log-CD case, we get

$$k^{\star} = \left[\frac{1-\alpha}{(1+n)(2+\rho)}\right]^{\frac{1}{1-\alpha}},$$

and thus have

$$f'(k^{\star}) = \alpha(k^{\star})^{\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(1+\rho),$$

which must be equal to n for the golden rule to hold. This is parameter dependent, and we might see both $k^{\star} < k^{gr}$ and $k^{\star} > k^{gr}$.[2] $k^{\star} > k^{gr} \Leftrightarrow f'(k^{\star}) < n = f'(k^{gr})$ is likely to be the case if agents are patient (ρ is small), low capital income share (α is small), or low population growth (n is small). If this is the case, it is possible to permanently increase consumption and thus utility.

Pareto inefficiency occurs, as we implicitly assume an infinite number of agents, which goes against the FWT. The benevolent social planner would redistribute wealth in order to achieve Pareto-optimality by transferring wealth from the young today to the old today, and promising the young today a similar transfer! Thus, social security is born, as a mean to address the potential welfare loss and ensure Pareto optimalism. It is important to note that social security might distort incentives not captured by the model.

Government expenditure When the government levies a tax on the young and makes "useless" purchases, capital accumulation falls. The analysis merely entail $\tilde{w}_t = w_t - T_t$ from before and, unlike the Ramsey model, shifts the capital accumulation schedule down.

Anticipated shocks will have no effect, while unanticipated temporary shocks will have similar effects to the long run shocks, as generations wither. Extending the life-span of individuals in the economy will make OLG dynamics behave similarly to the Ramsey model.

Ricardian equivalence This does not hold in the OLG, as agents have limited time horizons. Temporary shifts in financial policy (stimulus, tax hikes etc.) are seen as permanent from a household perspective, as the government budget constraint is not necessarily imposed in their lifetime, and thus correspondingly alters behaviour.

[2] Opposed to the Ramsey model, where $k^{\star} < k^{gr}$ always holds.

Fully funded

In a fully funded social security system, the government raises contributions from the current young, invests them, and then pays benefits to the contributors, when they are old.

Budget constraint The corresponding budget constraints are

$$c_{1t} + s_t + d_t = w_t \quad \text{and} \quad c_{2t+1} = (1 + r_{t+1} - \delta)s_t + b_{t+1} = (1 + r_{t+1} - \delta)(s_t + b_t)$$

with the Euler equation of

$$u'\left[w_t - (s_t + d_t)\right] = \frac{1 + r_{t+1} - \delta}{1 + \rho} u'\left[(1 + r_{t+1} - \delta)(s_t + d_t)\right]$$

The savings schedule is

$$s_t + b_t = (1 + n)k_{t+1}.$$

When the return on payments is assumed to be equal to the individual savings return, public saving exactly offsets the decline in private savings caused by the tax, as it too is being invested in capital and generate an equivalent return. Thus, we see dynamic efficiency under the same assumptions as the OLG without social security. If return for the social security system is not equivalent to that of externally saved capital, we would observe different results[3].

Pay-as-you-go

In the PAYG system, the government raises contributions from the current young and pays them out to the current old, such that $b_t = (1 + n)d_t$.

Budget constraint The budget constraints are now

$$c_{1t} + s_t + d_t = w_t \quad \text{and} \quad c_{2t+1} = (1 + r_{t+1} - \delta)s_t + (1 + n)d_{t+1},$$

with the Euler equation of

$$u'\left[w_t - (s_t + d_t)\right] = \frac{1 + r_{t+1} - \delta}{1 + \rho} u'\left[(1 + r_{t+1} - \delta)s_t + (1 + n)d_{t+1})\right]$$

The savings schedule is

$$s_t = (1 + n)k_{t+1}.$$

[3] If a certain amount is forcible saved to a lower return, then public savings do not 1:1 crowd out private saving. Aggregate dynamics are parameter dependent.

As the contributions are not intertemporally transferred, they do not influence the savings schedule.

The households now save less, but, unlike the fully funded system, this private dissaving is not offset by public saving, due to the non-intertemporality of the system. Young agent savings decrease, as disposable income is reduced and they are assured benefits when old.

Dynamic efficiency A system is dynamically inefficient if $1 + r_{t+1} - \delta < n + 1 \Leftrightarrow r_{t+1} < n + \delta$ holds. The condition states the return on public savings as larger than the return on private savings. Implementing a PAYG system can raise welfare for all agents, if the condition holds before the implementation (private capital accumulation is too high). If $r_{t+1} > n + \delta$ is true, then the system is dynamically efficient and implementing social security renders a suboptimal solution.

Government debt Introducing government debt, the government budget constraint of

$$T_t + (1 + n)b_{t+1} = (1 + r_t)b_t$$

follows, with the corresponding savings schedule of

$$s_t = (1 + n)(k_{t+1} + b_{t+1})$$

as savings is now split between capital and bonds. Capital accumulation is now smaller, as bonds replace capital and bonds need taxes to cover interest payments.

Under some assumptions, there is an equivalence between introducing government debt and a PAYG social security system. Government debt can have the same capital accumulation decrease effects as a PAYG system, as social security lowers savings due to contributions and benefits and debt crowds out capital.

Real business cycle theory

The model RBC explains drivers of business cycles through a micro-founded rational expectations framework. It focuses on productivity shocks and unifies growth theory with business cycle theory. The model is calibrated rather than derived.

Equilibrium Equilibrium is defined by characterizing a set of properties regarding the persistence and comovements of economic variables. Technological shocks (uncertainty in the growth rate of technology) act as the main propagation mechanism and the degree of transfer is determined by intertemporal substitution effects.

Supply side The combination of labour and capital produces a homogeneous product

$$Y_t = A_t F(K_t, N_t),$$

where A_t is technology, K_t is capital, N_t is labour input, and F satisfies the Inada conditions. Factor productivity is assumed to follow a secular path of

$$X_t = \gamma X_{t-1}, \quad \gamma > 1$$

and A_t growing as a unit root with a time trend as

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t.$$

Factor prices are determined by their marginal products. Capital stock evolves according to

$$K_{t+1} = I_t + (1-\delta)K_t, \quad \delta \in (0,1)$$

and output is limited by the resource constraint of

$$Y_t = C_t + I_t \Leftrightarrow I_t = Y_t - C_t$$

Demand side Rational expectations are assumed. Households are assumed to own the companies. Individuals split their time, given as H, between leisure, L_t, and labour, N_t

$$H = L_t + N_t,$$

with H normalized to 1. Household utility depending on leisure such that

$$\max E_0[\sum_{t=0}^{\infty} b^t U(C_t, L_t)].$$

with b serving as a preliminary discount factor.

Balanced growth path Imposing the BGP by scaling all variables to their growth rate such that $c_t = C_t/X_t$ etc. and ensuring factor market clearing yields the following optimization problems and solutions for the households

$$\max_{c_t,N_t,k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, L_t\right), \quad \beta = b\gamma^{1-\sigma}$$

$$\text{s.t.} \quad c_t + \gamma k_{t+1} = w_t N_t + \left(1 + r_t - \delta\right) k_t + \Pi_t$$

$$\Rightarrow \mathcal{L}_t = E_0 \sum_{t=0}^{\infty} [\beta^t u\left(c_t, L_t\right) + \lambda_t (w_t N_t + \left(1 + r_t - \delta\right) k_t + \Pi_t - \left(c_t + \gamma k_{t+1}\right)], \quad \lambda_t = u_c'(c_t, L_t)$$

$$\Rightarrow w_t = \frac{u_L'\left(c_t, L_t\right)}{u_c'\left(c_t, L_t\right)}$$

$$\beta E_t\left[\left(1 + r_{t+1} - \delta\right) u_c'\left(c_{t+1}, L_{t+1}\right)\right] = \gamma u_c'\left(c_t, L_t\right).$$

This leads to the Euler equation of

$$\frac{u_L'\left(c_{t+1}, L_{t+1}\right)}{u_L'\left(c_t, L_t\right)} = \frac{1}{\beta\left(1 + r_{t+1} - \delta\right)} \frac{w_{t+1}}{w_t}.$$

Under separable preferences, the Euler equation can be derived for each input separately. For the intertemporal elasticity of labour supply[4] under separate preferences it holds that if $\frac{w_{t+1}}{w_t} \uparrow$ then $\frac{L_{t+1}}{L_t} \downarrow$ when the substitution effect dominates the income effect and vice versa. The positive wealth (compensation) effect is seen from $r_{t+1} \uparrow$ leading to $\frac{L_{t+1}}{L_t} \uparrow$.

Under separable preferences, we see that

$$u'(c_t) = \beta E_t\left[\widetilde{R}_{t+1}\right] E_t\left[u_c'\left(c_{t+1}\right)\right] + \beta \operatorname{Cov}\left[\widetilde{R}_{t+1}, u_c'\left(c_{t+1}\right)\right],$$

where $\widetilde{R}_{t+1} = 1 + r_{t+1} - \delta$. This means that saving is attractive, when the gross interest rate is high, which decreases contemporary consumption.

Optimization for firms gives

$$\max_{N_t,k_t} \Pi_t = A_t F\left(k_t, N_t\right) - w_t N_t - R_t k_t$$

$$w_t = A_t F_N'(k_t, N_t) \quad \text{and} \quad r_t = A_t F_k'(k_t, N_t).$$

Imposing Walras' law yields market clearance.

Welfare The FWT applies such that the equilibrium is Pareto efficient.

Ricardian equivalence When introducing a government to the model, Ricardian equivalence holds, akin to the Ramsey model, as agents have infinite horizons.

Calibration The model is then parametrically calibrated such that key co-movements, autocorrelations etc.

[4]See Romer p. 198-199 for an improved explanation.

follow empirical observations.

Main implications The RBC model implies that business cycles are optimal responses to technological shocks. Thus, business cycles represent the time-varying Pareto optimal state. Policy interference in the real economy is difficult, costly, and likely to yield unwanted results.

Criticisms For the model to be correctly calibrated, an unrealistic intertemporal labour substitution rate (wage elasticity) is required. The implication of strong procyclical real wage movements and the Euler equation's incompatibility with the Equity Premium Puzzle are counterfactual. Lastly, to solely rely on large persistent technology shocks may seem unrealistic.

Extensions Some extensions measures to address criticisms of the main model are employment variability, labour supply shocks, and capacity utilization shocks.

Nominal rigidities (New Keynesian models)

General overview Models within this framework seeks to explain when money is non-neutral and the central bank's role. The key assumptions are non-flexible price and wage setting, thus introducing market imperfections, and involuntary unemployment. They focus primarily on the short run.

Market imperfections They might stem from market power imperfections or information asymmetries or frictions.

Monopolistic competition This is a necessity in order to introduce price rigidity and money non-neutrality. It can be on the supply side, demand side, or both. The following properties apply to a market under monopolistic competition:

- A large number of given firms and differentiated products exist.
- Each firm is a price maker of its own good, which is imperfectly substituteable.
- Price changes in one firm only affects demand for another firm minusculely.
- The short run competitive equilibrium defines prices and quantities such that supply equals demand and firm profits' are maximized given demand and other firms' prices.

This leads to Pareto inefficiency with an underutilization of resources. If price rigidity is introduced, for instance through menu costs, money are non-neutral.

Rational expectations Expectations to interest rate movements etc. are rational, when they are formed by knowing the structure of the economy and using all available information,

$$E_t[X_{t+1}] = E[X_{t+1}|I_t].$$

This excludes systematic forecast errors and the public can only be "fooled" once by unexpected actions.

Expectational differential equations When the distribution of an endogenous variable, y_t, is a function of the distribution of a know variable, x_t, such as

$$y_t = aE_t[y_{t+1}] + cx_t \Rightarrow y_t = c\sum_{i=0}^{\infty} a^i E_t[x_{t+i}], \quad \lim_{t\to\infty} a^T E_t[y_{t+T}] = 0, \quad \Delta x_t < \frac{1}{a} - 1,$$

through recursive substitution.

Blanchard-Kiyotaki model

This static new Keynesian model, examines monopolistic competition and shows the implications thereof, such as the non-neutrality of money in case of price adjustment costs.

Basic setup We consider a model with m firms and goods, each good being imperfectly substituteable, with one representative household, where money acts as numeraire and is accumulated. This model only considers monopolistic competition for firms but is expandable such that monopolistic competition in the wage/labour market is also present.

The representative household The representative household optimizes its utility over consumption, C, real money, M/P, and leisure, N,

$$\max_{C_i, N, \frac{M}{P}} U = C^{\gamma}\left(\frac{M}{P}\right)^{1-\gamma} - \frac{1}{\beta}N^{\beta}, \quad 0 < \gamma < 1, \quad \beta > 1,$$

subject to the resource constraint of

$$\sum_{i=1}^{m} P_i C_i + M = M_0 + WN + \sum_{i=1}^{m} \Pi_i$$

where the LHS is the household demand of money and value goods and the RHS is the initial endowment of money, total wages, and firm profits, which equals total household wealth. We implicitly assume that goods are non-transferable such that money is the only way to intertemporally transfer wealth.

Consumption and prices are given as

$$C = m^{\frac{1}{1-\theta}}\left(\sum_{i=1}^{m} C_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

and

$$P = \left(\frac{1}{m}\sum_{i=1}^{m} P_i^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

This yields a Lagrangian of

$$\max_{C_i,N,\frac{M}{P}} \mathcal{L} = \left[m^{\frac{1}{1-\theta}} \left(\sum_{i=1}^{m} C_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right]^{\gamma} \left(\frac{M}{P} \right)^{1-\gamma} - \frac{1}{\beta} N^{\beta} - \frac{\lambda}{P} \left(\sum_{i=1}^{m} P_i C_i + M - M_0 - WN - \sum_{i=1}^{m} \Pi_i \right)$$

with the corresponding F.O.C.'s of

$$\begin{aligned}
\frac{\partial U}{\partial C_i} &= 0 \Rightarrow \gamma \left(\frac{M}{PC} \right)^{1-\gamma} \left(\frac{C}{mC_i} \right)^{\frac{1}{\theta}} = \lambda \frac{P_i}{P} \\
\frac{\partial U}{\partial (M/P)} &= 0 \Rightarrow (1-\gamma) \left(\frac{M}{PC} \right)^{-\gamma} = \frac{\lambda}{P} \\
\frac{\partial U}{\partial N} &= 0 \Rightarrow N^{\beta-1} = \lambda \frac{W}{P}
\end{aligned}$$

Combining the two first yield the demand for good i.

$$\frac{P_i}{P} = \frac{\gamma}{1-\gamma} \frac{M}{PC} \left(\frac{C}{mC_i} \right)^{\frac{1}{\theta}}$$

From F.O.C. 2 and 3, aggregate labour supply can be derived as

$$\begin{aligned}
N^{\beta-1} &= \frac{W}{P}(1-\gamma) \left(\frac{M}{PC} \right)^{-\gamma} \\
\Rightarrow N^S &= [(1-\gamma)^{1-\gamma}\gamma^{\gamma}]^{\frac{1}{\beta-1}} \left(\frac{W}{P} \right)^{\frac{1}{\beta-1}}.
\end{aligned}$$

Plugging the demand schedule for goods into the definition for P yields

$$P = \frac{\gamma}{1-\gamma} \frac{M}{C},$$

which gives

$$C_i = \left(\frac{P_i}{P} \right)^{-\theta} \frac{C}{m}.$$

The demand for consumption and money as a function of endowment is then

$$\sum_{i=1}^{m} P_i C_i + M = I$$

while real consumption expenditure is

$$\sum_{i=1}^{m} \frac{P_i}{P} C_i = C \Leftrightarrow PC = \sum_{i=1}^{m} P_i C_i.$$

We then get aggregate consumption, output, and money as

$$C = \gamma \frac{I}{P}$$
$$M = (1-\gamma)I$$
$$C = Y = \frac{\gamma}{1-\gamma}\frac{M}{P}.$$

Firm optimization For firm i the optimization problem over the vector $\{P_i/P, Y_i, N_i\}_{i=1}^N$ is

$$\max_{\frac{P_i}{P}, N_i, Y_i} \Pi_i = P_i Y_i - W N_i$$

subject to the demand constraint of

$$Y_i = C_i = \left(\frac{P_i}{P}\right)^{-\theta} \frac{C}{m}$$

and input constraint w.r.t. labour of

$$Y_i = N_i^{\alpha}, \quad 0 < \alpha < 1.$$

Substitution yields the following optimization problem w.r.t. firm prices

$$\max_{P_i} \frac{P_i}{P}\left(\frac{P_i}{P}\right)^{-\theta}\frac{C}{m} - \frac{W}{P}\left[\left(\frac{P_i}{P}\right)^{-\theta}\frac{C}{m}\right]^{\frac{1}{\alpha}}$$

with the F.O.C. of

$$\frac{\partial \Pi_i}{\partial P_i} = 0 \Leftrightarrow \frac{P_i}{P} = \left[\frac{\theta}{\theta-1}\frac{1}{\alpha}\frac{W}{P}\left(\frac{C}{m}\right)^{\frac{1-\alpha}{\alpha}}\right]^{\frac{\alpha}{\alpha+\theta(1-\alpha)}}.$$

Imposing price homogeneity, $P_i = P$, for all firms and firm homogeneity, $C_i = Y_i = \frac{C}{m}$, aggregate demand for labour can be found as

$$N^D = \sum_{i=1}^{m} N_i^D = m Y_i^{\frac{1}{\alpha}} = m\left(\frac{C}{m}\right)^{\frac{1}{\alpha}} = m^{\frac{\alpha-1}{\alpha}}\left(\frac{\gamma}{1-\gamma}\frac{M}{P}\right)^{\frac{1}{\alpha}}$$

through the input resource constraint.

Labour market equilibrium This is found by setting $N^S = N^D$ as so:

$$N^S = N^D \Leftrightarrow \left[(1-\gamma)^{1-\gamma}\gamma^{\gamma}\right]^{\frac{1}{\beta-1}} \frac{W}{P}^{\frac{1}{\beta-1}} = m^{\frac{\alpha-1}{\alpha}}\left(\frac{\gamma}{1-\gamma}\frac{M}{P}\right)^{\frac{1}{\alpha}}.$$

Rearranging and taking logs such that real wages are a positive function of the real money supply yields the final equilibrium relationship:

$$\log \frac{W}{P} = \log \left[\frac{m^{\frac{(\alpha-1)|\beta-1}{\alpha}}}{1-\gamma} \left(\frac{\gamma}{1-\gamma} \right)^{\frac{\beta-1}{\alpha}-\gamma} \right] + \frac{\beta-1}{\alpha} \log \frac{M}{P}.$$

Goods market equilibrium Imposing symmetry in the production sector, $P_i = P$, the price setting rule from the firms maximization problem can be combined with the fact that $Y = C$ to yield

$$\frac{W}{P} = \frac{\theta-1}{\theta} \alpha \left(\frac{\gamma}{1-\gamma} \frac{M}{mP} \right)^{\frac{\alpha-1}{\alpha}},$$

which can be expressed in terms of mark-up pricing

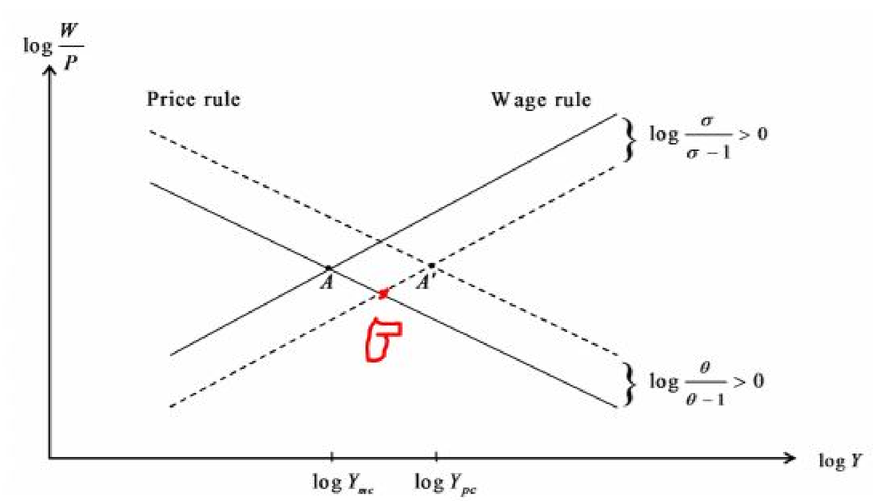
$$P = \frac{\theta}{\theta-1} MC, \quad MC = \frac{W}{\alpha} \left(\frac{\gamma}{1-\gamma} \frac{M}{mP} \right)^{\frac{1-\alpha}{\alpha}}.$$

The mark-up disappears, when monopolistic competition vanishes, $\theta \to \infty$, which eliminates the mark-up factor $\frac{\theta}{\theta-1}$.

Taking logs, we find real wages as a negative function of real money balances:

$$\log \frac{W}{P} = \log \left[\frac{\theta-1}{\theta} \alpha \left(\frac{\gamma}{1-\gamma} \frac{1}{m} \right)^{\frac{\alpha-1}{\alpha}} \right] - \frac{1-\alpha}{\alpha} \log \frac{M}{P}.$$

General equilibrium From here, the general equilibrium for real wages and real money balances can be found, as illustrated below with B representing the equilibrium in case of solely the firms operating within monopolistic competition.



Thus, we see an underutilization of resources as unemployment exists in equilibrium due to the monopolistic nature of firms. The Pareto-inferior underemployment that arise under the market power of monopolistic competition is an example of coordination failure. If we augment the model with household supplying labour monopolistically, $N = N_i$, with some coefficient, σ, determining the degree of monopoly, market imperfections will be larger. Total output lost through market power can be calculated as $\frac{Y}{Y_m}$ with Y corresponding to the equilibrium when $\frac{\theta}{\theta-1} = 0$.

Endogenous aggregate pricing If firms influenced and took into consideration their feedback price change effects to the general price level, such that $P = P(P_1, P_2, \dots P_m)$, welfare would improve, as individual price changes would be smaller. From this we see that coordination will yield larger total societal benefit.

Money non-neutrality For M to influence the real variables of the system, we need firms to incur a cost to shift prices (menu costs).

Menu cost theory Menu costs are modelled as fixed costs of price changes (for instance opportunity costs, information gathering, customer retention costs etc.).

The general price level is given as P^G. If $P^G \neq E[P^G]$, the monopolist's nominal per unit pricing, p is suboptimal, and profits can be increased by changing prices. If there is a cost, c, associated with this, prices should be changed if $E[\Pi] - c > \Pi$. If not, the monopolist will hold prices, p, which will lead to P^G having a real influence, as optimal monopoly quantity and price is not achieved.

Aggregating this effect m times in correspondence with the Blanchard-Kiyotaki model, money is non-neutral and can have a large impact on unemployment, output etc. If no firms respond to an increase in P^G due to menu costs, aggregate output would rise due to an aggregate nominal demand increase, as a function of the increased money supply, not being met with price hikes.

When menu costs are operative in both output and labor markets, output and employment adjust to demand, while prices and wages are unchanged. This is seen from the general expression for Y's dependency of M,

$$Y = \frac{\gamma}{1-\gamma}\frac{M}{P},$$

when assuming P is predetermined due to menu costs.[5]

Lucas model

The Lucas model assumes rational expectations and that producers act competitively. It implies that the central bank cannot use inflation to boost output if it does not have an information advantage (agents do not suffer from the "money illusion").

[5]More on BK and menu costs: http://web.econ.ku.dk/okocg/VM/VM-general/Kapitler%20til%20bog/Ch20-2015-2.pdf

Firm behaviour Firms determine their production quantity under uncertainty, as it cannot perfectly observe if price changes are relative (adjust production to meet demand shift) or general (do not adjust production). Thus, in case of a price increase, optimal response dictates a production increase of some amount, reflecting the probability that the price increase is a result of an increased demand.

Optimization problem Each household is assumed to produce and consume at the same time and optimizes

$$\max_{C_i, L_i} U_i = C_i - \frac{1}{\gamma} L_i^{\gamma}$$

w.r.t.

$$L_i = Y_i \quad \text{and} \quad C_i = \frac{P_i}{P} Y_i.$$

Combining this and taking the F.O.C. gives

$$\max_{Y_i} U_i = \frac{P_i}{P} Y_i - \frac{1}{\gamma} Y_i^{\gamma}$$
$$\Rightarrow \frac{P_i}{P} = Y_i^{\gamma-1} \Leftrightarrow Y_i = \left(\frac{P_i}{P}\right)^{\frac{1}{\gamma-1}}.$$

Taking logs yield the optimal production of

$$y_i = \frac{1}{\gamma - 1}(p_i - p).$$

The good specific demand curve is

$$y_i = y + z_i - \eta(p_i - p) = m - p + z_i - \eta(p_i - p),$$

where aggregate demand is $y = m - p$, $\eta > 0$, and z_i is a good specific demand shock.

Optimal production/supply curve Defining relative prices, $r_i = p_i - p$ and the observed prices, $p_i = p + (p_i - p) = p + r_i$, producers must infer observed price changes from general price changes or relative price changes. Thus, firm production is determined by

$$y_i = \frac{1}{\gamma - 1} E[r_i | p_i],$$

and the firm maximizes this under a certainty equivalence, as firms do not consider forecast errors and uncertainty.

Assuming $m \sim N(E[m], V_m)$ and $z_i \sim N(0, V_z)$, the expected relative price level can be determined from the observed prices as

$$E\left[r_i|p_i\right] = E\left[r_i\right] + \frac{V_r}{V_r + V_p}\left(p_i - E[p]\right) = \frac{V_r}{V_r + V_p}\left(p_i - E[p]\right),$$

which implies p and r_i are independent normally distributed by using the following rule:

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} E[x] \\ E[y] \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right) \to E[x|y] = E[x] + \frac{\Sigma_{11}}{\Sigma_{22}}(y - E[y]).$$

Inserting this signal extraction result in y_i gives optimal production for each firm

$$y_i = \frac{1}{\gamma - 1}\frac{V_r}{V_r + V_p}\left(p_i - E[p]\right) \equiv b\left(p_i - E[p]\right),$$

which can be aggregated across all firms

$$y = b(p - E[p]),$$

when assuming homogeneity, to the Lucas supply curve.

Equilibrium Setting supply equal to demand, $b(p - E[p]) = m - p$, yields the equilibrium of

$$p = \frac{1}{1+b}m + \frac{b}{1+b}E[p] = E[m] + \frac{1}{1+b}(m - E[m])$$
$$y = \frac{b}{1+b}m - \frac{b}{1+b}E[p] = \frac{b}{1+b}(m - E[m])$$

as $E[p] = E[m]$.

To solve for r_i, we take optimal production for the individual firm, $y_i = b(p_i - p) + b(p - E[p])$, and combine with individual good demand, $y_i = m - p + z_i - \eta(p_i - p)$:

$$b(p_i - p) + b(p - E[p]) = m - p + z_i - \eta(p_i - p)$$
$$\Leftrightarrow br_i + b(p - E[p]) == m - p + z_i - \eta r_i$$
$$\Leftrightarrow (b + \eta)r_i = m - p - +b(p - E[p]) + z_i = 0 + z_i$$
$$\Leftrightarrow r_i = \frac{z_i}{b + \eta}$$
$$\Rightarrow E[r_i] = 0 \quad \text{and} \quad V(r_i) = \frac{V_z}{(b + \eta)^2}.$$

b can be found as a function of parameters and exogenous variables as

$$b = \frac{1}{\gamma - 1}\left[\frac{V_z}{V_z + \frac{(1+\eta)^2}{(1+b)^2}V_m}\right],$$

which completes the model. It can be shown that

$$\frac{\partial b}{\partial V_z} > 0, \text{ and } \frac{\partial b}{\partial V_m} < 0.$$

b is the relative weight on output disturbances in relation to price in case of unexpected shocks, as it is the multiplier which changes in the relative prices carry over to production. This is rising in individual variance good demand shocks and decreasing in variance money supply shocks.

Implications If m rises unexpectedly, suppliers make some adjustments to their prices. Thus, both y and p is affected and output is raised. If m rises expectedly, the full effect of the shock is absorbed by price hikes.

The Phillips curve Assuming m is a random walk with drift, $m_t = m_{t-1} + c + u_t$, means that $E[m_t] = m_{t-1} + c$. The equilibrium equations are thus

$$p_t = E[m_t] + \frac{1}{1+b}(m_t - E[m_t]) = m_{t-1} + c + \frac{1}{1+b}u_t$$
$$y_t = \frac{b}{1+b}(m_t - E[m_t]) = \frac{b}{1+b}u_t.$$

Defining $\pi_t = p_t - p_{t-1}$, inserting, and rearranging yields the Lucas Phillips curve

$$\pi_t = m_{t-1} - m_{t-2} + \frac{1}{1+b}(u_t - u_{t-1}) = c + \frac{b}{1+b}u_{t-1} + \frac{1}{1+b}u_t \Leftrightarrow$$
$$\pi_t = E_{t-1}[\pi_t] + \frac{1}{b}y_t,$$

where inflation and output is positively correlated. Still, only unanticipated shocks have real output effects, as foreseen shocks simply result in a higher inflation.

Policy stabilization trade-off Monetary policy, such as increasing the growth rate of money, c, at time t, only has real effects if it is unanticipated, as this will lead to a temporarily higher unobserved money growth and thus output by firms is raised through the Lucas supply curve, stimulating the economy. Otherwise, the stimulus is entirely captured in price hikes. This means that there is no exploitable trade-off to policy makers to boost the economy indefinitely. Active monetary stabilization policy is not feasible in this model, as it can only take place in the event of superior information, which can be subject to costs akin to menu costs. A critique of the implications is that the model requires a large short-run elasticity of labour supply to generate unemployment, which is not found empirically.

The Lucas critique This states that when expectations influence equilibrium, then changes in policies will affect expectations. Thus, previously observed statistical relationships, such as the Phillips curve was subject to in the 70's, break down. In general, the critique states not to extrapolate the past into the future due to expectational influence on equilibrium.

General new Keynesian price setting framework

The general model considered is as follows, with households optimizing

$$U_i = C_i - \frac{1}{\gamma} L_i^\gamma \quad \text{s.t.} \quad C_i = \frac{P_i}{P} Y_i, \quad Y_i = L_i, \quad \text{and} \quad Y_i = \left(\frac{P_i}{P}\right)^{-\eta} Y$$

where $\gamma > 1$ and the last constraint signifies the demand for each good, i. The goods market has monopolistic competition.

Substitution and maximization yields

$$U_i = \left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} Y_i - \frac{1}{\gamma} Y_i^\gamma \Rightarrow$$
$$\frac{\partial U_i}{\partial Y_i} = 0 \Leftrightarrow Y_i = \left(1 - \frac{1}{\eta}\right)^{\frac{1}{\gamma-1}} \left(\frac{P_i}{P}\right)^{\frac{1}{\gamma-1}}$$

with the corresponding logarithmic desired individual price level of

$$\frac{1}{\gamma-1}(p_i^\star - p) = y_i + \mu, \quad \mu = -\frac{1}{\gamma-1} \log(1 - \frac{1}{\eta}).$$

Imposing homogeneity, $y = m - p$, ignoring the constant term, and setting $\phi = \gamma - 1$ gives the relative price level as an increasing function of aggregate demand

$$\frac{1}{\gamma-1}(p^\star - p) = y + \mu = m - p + \mu$$
$$\Leftrightarrow p^\star = \phi m + (1-\phi)p + \phi\mu \Rightarrow p_i - p = \phi(m - p).$$

ϕ measures real rigidity with an inverse relationship. If $\gamma \uparrow$, then labour elasticity decreases. Thus, aggregate movements in demand will have higher pass-through effects on price, as labour does not adjust. This means that real rigidity is low when ϕ is high, and vice versa.

Taking expectations and imposing homogeneity gives the equilibrium of

$$p_t = E[m_t | I_t]$$
$$y_t = m_t - E[m_t | I_t]$$

similarly to the Lucas model. Thus, for anticipated shocks to have real effects, frictions must be introduced.

Dynamic new Keynesian models

Fischer model If there are frictions in price settings, then anticipated shocks have real effects. If firms determine their prices, which can be different from one period to another, for two periods, then the price setting system looks as so

$$p_t^1 = E_{t-1}\left[p_t^*\right] = \phi E_{t-1}\left[m_t + v_t\right] + (1-\phi)\frac{1}{2}\left(p_t^1 + p_t^2\right)$$
$$p_t^2 = E_{t-2}\left[p_t^*\right] = \phi E_{t-2}\left[m_t + v_t\right] + (1-\phi)\frac{1}{2}\left(E_{t-2}\left[p_t^1\right] + p_t^2\right)$$

with

$$p_t = \frac{1}{2}\left(p_t^1 + p_t^2\right)$$
$$p_t^* = \phi(m_t + v_t) + (1-\phi)\frac{1}{2}\left(p_t^1 + p_t^2\right)$$

and

$$y_t = m_t - p_t + v_t,$$

where v_t is a demand shock.

Rearranging gives two price setting rules, where prices are expected to be the profit-maximizing prices of the previous period, of

$$p_t^2 = E_{t-2}[m_t + v_t]$$
$$p_t^1 = E_{t-2}[m_t + v_t] + \frac{2\phi}{1+\phi}(E_{t-1}[m_t + v_t] - E_{t-2}[m_t + v_t])$$

with equilibrium given as

$$p_t = E_{t-2}\left[m_t + v_t\right] + \frac{\phi}{1+\phi}\left(E_{t-1}\left[m_t + v_t\right] - E_{t-2}\left[m_t + v_t\right]\right)$$
$$y_t = \left(m_t + v_t - E_{t-1}\left[m_t + v_t\right]\right) + \frac{1}{1+\phi}\left(E_{t-1}\left[m_t + v_t\right] - E_{t-2}\left[m_t + v_t\right]\right).$$

Unanticipated shocks have real effects, which is shown from the first term in y. Anticipated shocks also have real effects, due to price inflexibility in the short run. If prices are more responsive ($\phi \uparrow$ such that real rigidity is low), the real effect of anticipated is small. If there is no real rigidity and no unanticipated shocks, there is no real impact of monetary shocks to real variables.

Monetary policy If monetary policy follows the rule of

$$m_t = a_1\epsilon_{t-1} + a_2\epsilon_{t-2} + \dots + a_n\epsilon_{t-n} + \dots$$

and v_t is a random walk, then

$$m_t + v_t = a_1\epsilon_{t-1} + a_2\epsilon_{t-2} + \dots + a_n\epsilon_{t-n} + \dots + v_{t-1} + \epsilon_t.$$

Taking expectations and inserting in equilibrium yields

$$y_t = \epsilon_t + \frac{1+a_1}{1+\phi}\epsilon_{t-1}.$$

Thus, optimal stabilization policy dictates that $a_1 = -1$. Other coefficients are irrelevant, as their corresponding shocks have been absorbed in the price level. Shocks are persistent for one period. The model and optimal policy rule is easily extended by adding price setting periods.

Taylor model In this modification of the Fischer model, prices are deterministically fixed and equivalent for three periods such that

$$p_t = \frac{1}{3}(x_t + x_{t-1} + x_{t-2})$$
$$x_t = \frac{1}{3}(p_t^\star + E_t[p_{t+1}^\star] + E_t[p_{t+2}^\star]).$$

Assuming $p_t^\star = m_t$ (not considering real rigidity) and that m_t follows a random walk, $m_t = m_{t-1} + \varepsilon_t$, the inflation schedule can be derived as

$$\pi_t = p_t - p_{t-1} = \frac{1}{3}(\varepsilon_t + \varepsilon_{t-1}\varepsilon_{t-2}).$$

Thus, inflation follows an MA(2) process and the economy exhibits some price level intertia.

Calvo model Here, prices are stochastic and fixed and firms might update prices with probability $0 < \alpha \leq 1$. The general price level is given as

$$p_t = \alpha x_t + (1-\alpha)p_{t-1},$$

where x_t is the price chosen by firms that can update their prices. Subtracting p_{t-1} yields an initial Phillips curve of

$$\pi_t = \alpha(x_t - p_{t-1}).$$

$x_t^\star$ is a weighted (discounted) average of all possible future optimal prices:

$$x_t = [(1-\beta(1-\alpha)]\sum_{j=0}^{\infty}\beta^j(1-\alpha)^j E_t\left[p_{t+j}^*\right]$$
$$\Leftrightarrow x_t = \left[(1-\beta(1-\alpha)]p_t^* + \left[(1-\beta(1-\alpha)]\sum_{j=1}^{\infty}\beta^j(1-\alpha)^j E_t\left[p_{t+j}^*\right]\right]\right].$$

From this, the new Keynesian Phillips curve can be derived as

$$\pi_t = \frac{\alpha}{1-\alpha}[1-\beta(1-\alpha)]\phi y_t + \beta E_t\left[\pi_{t+1}\right] = \frac{\alpha\phi}{1-\alpha}[1-\beta(1-\alpha)]\sum_{j=0}^{\infty}\beta^j E_t\left[y_{t+j}\right].$$

In opposition to the Lucas model, inflation is now a function of expected future inflation instead of expected current inflation. Inflation today thus reflects log-deviations for steady state output.

Persistence puzzle The NKPC fails to generate inflation persistence. Modifying the model with lagged inflation values, adaptive expectations, etc. and thereafter calibrating parameters addresses this issue and aids empirical fit.

The dynamic New Keynesian Model A model addressing the concerns risen with the persistence model. The demand side is given as

$$y_t = E_t[y_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}]$$

with the supply side being the NKPC

$$\pi_t = \kappa y_t + \beta E_t\left[\pi_{t+1}\right].$$

The policy rule closing the model is suggested as

$$i_t = \phi_\pi \pi_t + \phi_y y_t + v_t.$$

Monetary policy

The model We seek to understand the stabilizing prowess as well as the related costs and benefits of active monetary policy by examining different approaches to active monetary policy. Policy makers must satisfy an ex-post optimality constraint, taking expectations as given, while also considering future implications of active monetary policy and policy shifts. Policy makers have a positive and normative objective to satisfy, respectively how policy makers react to different incentives and, given the positive approach, how policy makers maximize utility.

Demand side We consider the reduced form demand equation of

$$\pi = m + v + \mu,$$

where m is money growth, v is a demand shock, and μ is the control error (the difference between desired and actual monetary policy).

Supply side We consider the NKPC of

$$x = \theta + (\pi - \pi^e) - \epsilon,$$

where x is output (or output growth), ϵ is a supply shock, and θ is the stochastic potential output ($\bar{x}$) or long term growth rate (g).

Monetary policy rule In order to bind the system together, we need a monetary policy rule that specifies m, and thus π. We will observe a trade-off between inflation and output stabilization. If there are no supply shocks, $\epsilon = 0$, there is no trade-off and we can perfectly stabilize output.

Commitment

Under commitment, the monetary policy rule is announced first, then θ is observed by all agents, π^e is formed, the shocks v and ϵ are realized, m is determined by the government, and lastly, μ is realized with π and x.

Both the private sector and the government observe θ but only the government observe ϵ and v. Thus, only unanticipated policy affects real variables as the government has an information advantage.

The social loss function The government seeks to minimize the social loss function, which is determined by expected deviations from optimality as

$$E[L(\pi, x)] = \frac{1}{2}E\left[(\pi - \bar{\pi})^2 + \lambda(x - \bar{x})^2\right].$$

Optimal ex-ante policy is the result of this minimization problem.

Policy rule If the social loss function is quadratic, optimal policies are always linear. The specified policy rule is

$$m = \psi + \psi_\theta \theta + \psi_v v + \psi_\epsilon \epsilon.$$

If the rule is credible, then expected inflation is

$$\pi^e = E[\pi|\theta] = E[m|\theta] + E[v|\theta] + E[\epsilon|\theta] = E[m|\theta] = \psi + \psi_\theta \theta.$$

Demand Taking expectations of the baseline and inserting, reveals demand as

$$x = \theta + m - E[m|\theta] + v + \mu - \epsilon.$$

Equilibrium Inserting the monetary policy rule and expected inflation in demand and supply functions gives

$$\pi = \psi + \psi_\theta \theta + (1 + \psi_v)v + \psi_\epsilon \epsilon + \mu$$
$$x = \theta + (1 + \psi_v)v + (-1 + \psi_\epsilon)\epsilon + \mu.$$

Optimal policy Plugging the equilibrium relationship into the social loss function, and remembering the cross-correlation uncorrelated terms, gives

$$\begin{aligned} E[L] &= \frac{1}{2}E\left[(\psi + \psi_\theta \theta + (1+\psi_v)\, v + \psi_\epsilon \epsilon + \mu - \bar{\pi})^2 + \lambda\left(\theta + (\psi_v + 1)\, v + (\psi_\epsilon - 1)\,\epsilon + \mu - \bar{x}\right)^2\right] \\ &= \frac{1}{2}E\left[\psi^2 + \bar{\pi}^2 - 2\psi\bar{\pi} + \psi_\theta^2 \sigma_\theta^2 + (1+\psi_v)^2 \sigma_v^2 + \psi_\varepsilon^2 \sigma_\epsilon^2 + \sigma_\mu^2 + \lambda\left(\bar{x}^2 + \sigma_\theta^2 + (1+\psi_v)^2 \sigma_v^2 + (\psi_\epsilon - 1)^2 \sigma_\epsilon^2 + \sigma_\mu^2\right)\right]. \end{aligned}$$

Optimal monetary rule is found by taking the F.O.C.'s for the coefficients.

1. $\frac{dE[L(\pi,x)]}{d\psi} = 0 \longrightarrow \psi = \bar{\pi}$
2. $\frac{dE[L(\pi,x)]}{d\psi_\theta} = 0 \longrightarrow \psi_\theta = 0$
3. $\frac{dE[L(\pi,x)]}{d\psi_v} = 0 \longrightarrow \psi_v = -1$
4. $\frac{dE[L(\pi,x)]}{d\psi_e} = 0 \longrightarrow \psi_\epsilon + \lambda\,(\psi_\epsilon - 1) = 0 \longrightarrow \psi_\epsilon = \frac{\lambda}{1+\lambda}$

The first and second rule anchors inflation, the third rule stabilizes demand shocks fully, and the fourth rule absorbs supply shocks in prices and output in correspondence to societal preferences.

The optimal rule is then

$$m = \bar{\pi} - v + \frac{\lambda}{1+\lambda}\epsilon$$

with the corresponding equilibrium of

$$\pi^C = \bar{\pi} + \frac{\lambda}{1+\lambda}\epsilon - \mu$$
$$x^C = \theta - \frac{1}{1+\lambda}\epsilon + \mu.$$

Since policy makers cannot control μ, this is disregarded.

Solution framework The simple step-by-step solution method to monetary policy under commitment is as follows:

1. Specify the supply and demand equations.
2. Specify the monetary policy rule for m.
3. Take expectations of this to obtain π^e.
4. Insert m and π^e in the demand and supply equations to obtain equilibrium.
5. Insert the equilibrium in the social loss function and minimize w.r.t. the reaction parameters of the monetary policy rule.
6. Insert results in equilibrium to obtain x and π in correspondence to societal preferences.

Discretion

Under discretion, the monetary policy rule is unspecified and the government chooses m at the last step. This means the information advantage, and ability to alter outcome, to the government is greater than under commitment. Policy choice thus takes place after uncertainty is realized.

Credibility The assumption of monetary policy taking place in a static condition is flawed. The repeated/sequential nature of monetary policy setting leads to credibility playing a large role, which discretion captures better.

Equilibrium satisfaction Policy must be ex-post optimal, $\frac{\partial L}{\partial m} = 0$ given π^e and θ, and expectations are rational. From this, equilibrium must a Nash equilibrium.

Supply and demand The nature of this system reveals supply as

$$x = \theta + (\pi - \pi^e) - \epsilon$$

and demand as

$$\pi = m.$$

Optimal rule Inserting supply in the SL, where expectations are dropped, gives optimal policy rule through the F.O.C. for π:

$$L(\pi, x) = \frac{1}{2}\left[(\pi - \bar{\pi})^2 + \lambda(x - \bar{x})^2\right] = \frac{1}{2}\left[(\pi - \bar{\pi})^2 + \lambda(\theta + (\pi - \pi^e) - \epsilon - \bar{x})^2\right]$$
$$\Rightarrow \frac{\partial L}{\partial \pi} = 0 \Leftrightarrow \pi - \bar{\pi} + \lambda(\theta + \pi - \pi^e - \epsilon - \bar{x}) = 0$$
$$\Leftrightarrow \pi = \frac{1}{1+\lambda}\bar{\pi} + \frac{\lambda}{1+\lambda}\left(-\theta + \pi^e + \epsilon + \bar{x}\right).$$

Assuming $\bar{x} > \theta$ means that distortions lead potential output below its desired level.

Equilibrium Private sector expectations can be found from the optimal rule such that

$$\pi^e = E[\pi|\theta] = \bar{\pi} + \lambda(\bar{\pi} - \theta).$$

This gives the following equilibrium

$$\pi^D = \bar{\pi} + \lambda(\bar{x} - \theta) + \frac{\lambda}{1+\lambda}\epsilon$$
$$x^D = x^C = \theta - \frac{1}{1+\lambda}\epsilon.$$

Output is equal to that under commitment, while inflation is larger with a margin of $\lambda(\bar{x} - \theta)$.

Inflation bias The term $\lambda(\bar{x} - \theta)$ is also known as the inflation bias of discretion. The bias rises with λ, which is the social relative weight on output stability. If the government tries, unsuccessfully, to boost the economy through surprise inflation, then the model predicts a positive correlation between inflation levels and its volatility (both depend on λ). This increases the general price volatility under discretion in comparison to commitment. The term stems from the lack of commitment and the temptation to temporarily boost the economy, which is rising in $\bar{x}$.

Solution framework To solve the model with monetary policy under discretion the steps are as follows:

1. Specify the supply and demand equations, where $\pi = m$.
2. Insert in social loss function and minimize w.r.t. π.
3. Take expectations of this expression to obtain $E[\pi]$.
4. Insert results in the supply and demand equations to obtain an equilibrium.

Reputation As the game is sequential and the government can only boost output by unanticipatedly distorting the equilibrium, a repeated loss evaluation is more correct:

$$E_t \sum_{j=0}^{\infty} \beta^j E\left[L\left(\pi_{t+j}, x_{t+j}\right)\right]$$

The government will then try to optimize over the repeated function rather than the static equilibrium, which lessens the incentive to unexpectedly perform active monetary policy.

Sequential equilibrium If the social loss function is simplified as

$$L_t(\pi, x) = \frac{\pi^2}{2} - \lambda x$$

and expectations follow the rule of

$$\pi_t^e = \left\{ \begin{array}{l} 0 \text{ if } \pi_v = \pi_v^e, \quad v = t-1, \dots, t-T \\ \lambda \text{ otherwise} \end{array} \right\},$$

then the government has the minimization problem of

$$\begin{aligned} \arg\min_{\pi_t} L_t &= \frac{\pi^2}{2} - \lambda x \\ \Rightarrow \pi_t &= \lambda. \end{aligned}$$

Deviation from monetary policy rule Deviations from the previously determined rule by taking advantage of pre-set private expectations are determined by evaluating the total discounted social loss against the total discounted social benefit.

Satisfying expectations lead to a loss of $L_t = -\lambda(\theta_t - \epsilon_t)$. Deviating from commitment leads to the aggregate benefit of

$$B = L(0, \theta_t - \epsilon_t) - L(\lambda, \lambda + \theta_t - \epsilon_t) = \frac{\lambda^2}{2}.$$

The loss will occur from time $t+1$ and forwards as

$$\begin{aligned} C &= E_s \sum_{t=s+1}^{T} \beta^{t-s} \left[L(\lambda, \theta_t - \epsilon_t) - L(0, \theta_t - \epsilon_t) \right] \\ &= E_s \sum_{t=s+1}^{T} \beta^{t-s} \left[\frac{\lambda^2}{2} - \lambda(\theta_t - \epsilon_t) + \lambda(\theta_t - \epsilon_t) \right]. \\ &= \beta \frac{1-\beta^T}{1-\beta} \frac{\lambda^2}{2} \end{aligned}$$

Deviations are then guided by considering and weighing the loss against the benefit.

$$B \leq C \Leftrightarrow 1 \leq \beta \frac{1-\beta^T}{1-\beta}$$

If the social loss function was quadratic, the decision would depend on θ as well.

Considering an infinite horizon aids explaining the long-term loss of credibility and reputation, resulting in unfavorable private inflation expectations, from deviating.

Pegged currency If a currency peg is credible, then $\pi = \pi^\star$. The presented trade-off is one between lower inflation against higher output volatility (theoretically!). Comparing discretionary monetary policy with a pegged currency yields

$$E\left[L\left(\pi^D, x^D\right)\right] - E\left[L\left(\pi^S, x^S\right)\right] = \frac{1}{2}\left[\lambda^2\left(\bar{x}^2 + \sigma_\theta^2 - \frac{1}{1+\lambda}\sigma_\epsilon^2\right) - \sigma_{\pi^*}^2\right],$$

which should be positive if a pegged currency is favorable. The first term including λ represents the gain from the peg that eliminates inflation bias. The second and third terms in the parentheses represent the loss of higher output volatility by having a peg as active monetary policy w.r.t. output stabilization is impossible in a fixed exchange-rate regime.

Exam - January 2017

Assignment 1

This solution has been modified with population growth in correspondence to Emiliano's "hint".

We consider an OLG economy, where identical firms maximize profit

$$\max_{K_t, L_t} AK_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t, \quad \alpha \in (0,1).$$

The gross return on savings is $R_t = 1 + r_t$ and we disregard δ. The government runs a PAYG system of social security.

a - Firm profit maximization The firm maximizes

$$\max_{K_t, L_t} AK_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t, \quad \alpha \in (0,1)$$

which yields the factor prices of

$$\frac{\partial \Pi_t}{\partial K_t} = 0 \Leftrightarrow R_t = \alpha A k_t^{\alpha-1}$$
$$\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = (1-\alpha) A k_t^\alpha.$$

b - The individuals problem of optimal intertemporal resource allocation This problem can be raised as

$$\max_{c_{1t},c_{2t+1}} \ln c_{1t} + \frac{1}{1+\rho}\ln c_{2t+1}$$
$$\text{subject to } (1-\tau)w_t = c_{1t} + s_t \text{ and } c_{2t+1} = (1+n)\tau w_{t+1} + s_t(1+r_{t+1}).$$

The lifetime budget constraint can be found as

$$c_{2t+1} = (1+n)\tau w_{t+1} + ((1-\tau)w_t - c_{1t})(1+r_{t+1}).$$

Combining this with the optimization problem yields the Lagrangian of

$$\mathcal{L}(c_{1t}, c_{2t+1}|\lambda) = \ln c_{1t} + \frac{1}{1+\rho}\ln c_{2t+1} + \lambda((1+n)\tau w_{t+1} + ((1-\tau)w_t - c_{1t})(1+r_{t+1}) - c_{2t+1}).$$

The F.O.C.'s are

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0 \Leftrightarrow \frac{1}{c_{1t}} = \lambda(1+r_{t+1})$$
$$\frac{\partial \mathcal{L}}{\partial c_{2t+1}} = 0 \Leftrightarrow \frac{1}{1+\rho}\frac{1}{c_{2t+1}} = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow c_{2t+1} = (1+n)\tau w_{t+1} + ((1-\tau)w_t - c_{1t})(1+r_{t+1}).$$

The first two F.O.C.'s combine into the Euler equation of

$$\frac{c_{2t+1}}{c_{1t}} = \frac{1+r_{t+1}}{1+\rho},$$

which states the intertemporal choice of consumption as negatively growing in the discount rate, ρ, and positively on the rate of return, $1+r_{t+1}$.

The budget constraints can be inserted in the Euler equation to find the savings schedule:

$$\frac{s_t(1+r_{t+1}) + (1+n)\tau w_{t+1}}{(1-\tau)w_t - s_t} = \frac{1+r_{t+1}}{1+\rho} \Leftrightarrow s_t(1+r_{t+1}) + (1+n)\tau w_{t+1} = \frac{1+r_{t+1}}{1+\rho}((1-\tau)w_t - s_t)$$
$$\Leftrightarrow s_t((1+r_{t+1}) + \frac{1+r_{t+1}}{1+\rho}) = \frac{1+r_{t+1}}{1+\rho}(1-\tau)w_t - (1+n)\tau w_{t+1}$$
$$\Leftrightarrow s_t\frac{(1+r_{t+1})(1+\rho) + 1 + r_{t+1}}{1+\rho} = s_t\frac{(2+\rho)(1+r_{t+1})}{1+\rho} = \frac{1+r_{t+1}}{1+\rho}(1-\tau)w_t - (1+n)\tau w_{t+1}$$
$$\Leftrightarrow s_t = \frac{1}{2+\rho}(1-\tau)w_t - \frac{1+\rho}{2+\rho}\frac{1+n}{1+r_{t+1}}\tau w_{t+1}.$$

The same result could have been obtained by direct substitution of s_t into the utility maximization problem:

$$\begin{aligned}
&\max_{s_t} \ln{(1-\tau)w_t - s_t} + \frac{1}{1+\rho}\ln{(1+n)\tau w_{t+1} + s_t(1+r_{t+1})} \Rightarrow \frac{\partial}{\partial s_t} = 0 \\
&\Leftrightarrow \frac{1}{(1-\tau)w_t - s_t} = \frac{1+r_t}{1+\rho}\frac{1}{(1+n)\tau w_{t+1} + s_t(1+r_{t+1})} \\
&\Leftrightarrow s_t(1+r_{t+1}) + (1+n)\tau w_{t+1} = \frac{1+r_{t+1}}{1+\rho}((1-\tau)w_t - s_t) \\
&\Leftrightarrow s_t = \frac{1}{2+\rho}(1-\tau)w_t - \frac{1+\rho}{2+\rho}\frac{1+n}{1+r_{t+1}}\tau w_{t+1}.
\end{aligned}$$

c - Deriving the capital accumulation schedule This is derived from the basic form of $(1+n)k_{t+1} = s_t$ by inserting the functional forms of r_{t+1}, w_t, and w_{t+1} as

$$\begin{aligned}
(1+n)k_{t+1} &= \frac{1}{2+\rho}(1-\tau)(1-\alpha)Ak_t^\alpha - \frac{1+\rho}{2+\rho}\frac{1+n}{\alpha Ak_{t+1}^{\alpha-1}}\tau(1-\alpha)Ak_{t+1}^\alpha \\
\Leftrightarrow (1+n)k_{t+1} &= \frac{1}{2+\rho}(1-\tau)(1-\alpha)Ak_t^\alpha - \frac{1+\rho}{2+\rho}\frac{1+n}{\alpha}(1-\alpha)\tau k_{t+1} \\
\Leftrightarrow k_{t+1}\left((1+n) + \frac{1+\rho}{2+\rho}\frac{1+n}{\alpha}(1-\alpha)\tau\right) &= \frac{1}{2+\rho}(1-\tau)(1-\alpha)Ak_t^\alpha \\
\Leftrightarrow k_{t+1}\left(\frac{(1+n)(2+\rho)\alpha + \tau(1+n)(1+\rho)(1-\alpha)}{(2+\rho)\alpha}\right) &= \frac{1}{2+\rho}(1-\tau)(1-\alpha)Ak_t^\alpha \\
\Leftrightarrow k_{t+1} &= \left(\frac{(1-\tau)(1-\alpha)\alpha A}{(1+n)(2+\rho)\alpha + \tau(1+n)(1+\rho)(1-\alpha)}\right)k_t^\alpha.
\end{aligned}$$

We see that the capital accumulation schedule satisfies the Inada conditions, which points towards a stable steady state. We see two capital depressing accumulation effects from τ, one in the numerator from lowered income when young, and one in the denominator from one receiving benefits when old.

d - Deriving steady state level of capital This is done by imposing capital steady state, $k_t = k_{t+1} = k$ such that

$$k = \left(\frac{(1-\tau)(1-\alpha)\alpha A}{(1+n)(2+\rho)\alpha + (1+n)(1+\rho)(1-\alpha)\tau}\right)k^\alpha \Leftrightarrow k^\star = \left(\frac{(1-\tau)(1-\alpha)\alpha A}{(1+n)(2+\rho)\alpha + (1+n)(1+\rho)(1-\alpha)\tau}\right)^{\frac{1}{1-\alpha}}.$$

The two tax-effects of the PAYG system is still visibly present in the steady state.

e - The dismantling of social security at time t=T It is important to note that this takes place after savings decisions has been made. Thus, capital accumulation at time $T+1$ is equal to the pre-dismantlement steady state as it is dependent on previous savings decisions.

The new steady state is derived by setting $\tau = 0$

$$k^{new} = \left(\frac{(1-\alpha)\alpha A}{(1+n)(2+\rho)\alpha}\right)^{\frac{1}{1-\alpha}}.$$

This is unquestionably larger than the previous steady state. As the new steady state is larger than the old one, capital at time $T+1$ is locked in by the old system due to savings decisions already having been made, and that the capital accumulation schedule is an increasing function of itself, it must hold that capital at time $T+2$ is larger than at time $T+1$.

f - Considering the old generation As savings decisions have already been made at time T, the young generation at that point has not had the chance to adjust (increase) their savings schedule to optimally correct for the dismantling of social security. This leads to sub-optimal savings decisions. Thus, consumption when old, c_{2T+1}, will be lower than otherwise, and the old are then worse off at time $T+1$.

g - Dynamic efficiency Dynamic inefficiency, also known as Pareto inefficiency, stems from the possibility of over-saving such that $k^\star > k^{gr}$, where k^{gr} is the consumption maximizing balanced growth path (the argument that maximizes $c = f(k) - nk$). Thus, the condition for dynamic efficiency can be expressed in the terms of marginal products as $r_t > n$, where the return on private savings is larger than on public savings. As r_t is decreasing in k_t and steady state of capital in the new dismantlement situation is larger than under the PAYG-system, it can be inferred that dynamic efficiency is more likely in the capital depressing case of the social security system still being in place. The dismantlement thus presents a risk of dynamic inefficiency.

Assignment 2

We consider a model of monetary policy where the government controls inflation directly, $m_t = \pi_t$, where m_t is the growth rate of the money supply.

The loss function is

$$L_t(\pi_t, x_t) = \frac{1}{2}(\pi_t^2 + \lambda(x_t - \bar{x})^2),$$

which we notice is quadratic, such that the optimal rule under commitment is linear in its parameters.

Supply is specified by the new Keynesian Phillips curve as

$$x_t = \theta_t + \pi_t - \pi_t^e,$$

where we immediately notice the lack of a supply shock, ϵ_t. This means that we will face no trade-off between inflation and output stability.

a - Optimal policy under commitment The linear optimal monetary policy rule is guessed to be in the shape of

$$m_t = \pi_t = \psi + \psi_\theta \theta_t.$$

From here, we already know that $\psi = 0$ in optimum, as there is no inflation target. We can also derive private sector inflation expectations as

$$E[m_t] = E[\pi_t] = \pi_t^e = E[\psi + \psi_\theta \theta_t] = \psi + \psi_\theta \theta_t,$$

as we observe θ prior to forming expectations.

This leads to an output of

$$x_t = \theta_t + \pi_t - \pi_t^e = \theta_t + (\psi + \psi_\theta \theta_t) - (\psi + \psi_\theta \theta_t)$$
$$\Leftrightarrow x_t^C = \theta_t.$$

Inserting this and the monetary policy rule in the loss function and taking expectations yield

$$E_t[L_t] = E_t\left[\frac{1}{2}((\psi + \psi_\theta \theta_t)^2 + \lambda(\theta_t - \bar{x})^2)\right] = \frac{1}{2}E\left[\psi^2 + \psi_\theta^2 \theta^2 + 2\psi\psi_\theta\theta_t + \lambda(\theta_t^2 + \bar{x}^2 - 2\theta_t \bar{x})\right].$$

Minimizing this w.r.t. the policy parameters gives the F.O.C.'s of

$$\frac{\partial E_t[L_t]}{\partial \psi} = 0 \Leftrightarrow \psi + \psi_\theta E[\theta_t] = 0$$
$$\frac{\partial E_t[L_t]}{\partial \psi_\theta} = 0 \Leftrightarrow \psi_\theta E[\theta_t^2] + \psi E[\theta_t] = 0.$$

From these, the minimizing solution of $\psi = \psi_\theta = 0$ is clear.

Thus, equilibrium under commitment is

$$x_t^C = \theta_t$$
$$\pi_t^C = 0.$$

b - Equilibrium under discretion Here, the government do not follow an explicitly stated rule, but instead minimizes the loss function directly over π_t:

$$\min_{\pi_t} L_t = \frac{1}{2}(\pi_t^2 + \lambda(\theta_t + \pi_t - \pi_t^e - \bar{x})^2)$$
$$\Rightarrow \frac{\partial L_t}{\partial \pi_t} = 0 \Leftrightarrow \pi_t = \lambda(\theta_t + \pi_t - \pi_t^e - \bar{x}) \Leftrightarrow \pi_t = \frac{\lambda}{1+\lambda}(\bar{x} + \pi_t^e - \theta_t).$$

From here, inflation expectations can be derived

$$E(\pi_t) = \pi_t^e = E\left[\frac{\lambda}{1+\lambda}(\bar{x} + \pi_t^e - \theta_t)\right]$$
$$\Leftrightarrow \pi_t^e\left(1 - \frac{\lambda}{1+\lambda}\right) = \pi_t^e\left(\frac{1}{1+\lambda}\right) = \left[\frac{\lambda}{1+\lambda}(\bar{x} - \theta_t)\right]$$
$$\Leftrightarrow \pi_t^e = \lambda(\bar{x} - \theta_t).$$

Inserting this in the optimal inflation response, yields the equilibrium inflation of

$$\pi_t^D = \frac{\lambda}{1+\lambda}(\bar{x} + \lambda(\bar{x} - \theta_t) - \theta_t) = \lambda(\bar{x} - \theta_t) = -\lambda(\theta_t - \bar{x}).$$

Equilibrium output is thus

$$x_t^D = \theta_t + \pi_t - \pi_t^e = \theta_t.$$

The condition $\bar{x} > \theta_t$ ensures the presence of an inflation bias. This stems from the combination of a lack of commitment and the temptation to raise output temporarily through creating inflation. It is rising in $\bar{x}$, as a higher policy output target will increase this temptation to stimulate the economy.

c - Equilibrium under discretion when expectations are formed before potential output is realized When this is the case, the central bank still minimizes the loss function w.r.t. π_t such that

$$\min_{\pi_t} L_t = \frac{1}{2}(\pi_t^2 + \lambda(\theta_t + \pi_t - \pi_t^e - \bar{x})^2)$$
$$\Rightarrow \frac{\partial L_t}{\partial \pi_t} = 0 \Leftrightarrow \pi_t = \lambda(\theta_t + \pi_t - \pi_t^e - \bar{x}) \Leftrightarrow \pi_t = \frac{\lambda}{1+\lambda}(\bar{x} + \pi_t^e - \theta_t).$$

The private sector expectations now differ, as it is assumed that $E_t[\theta_t] = 0$. Thus,

$$E(\pi_t) = \pi_t^e = E\left[\frac{\lambda}{1+\lambda}(\bar{x} + \pi_t^e - \theta_t)\right]$$
$$\Leftrightarrow \pi_t^e\left(1 - \frac{\lambda}{1+\lambda}\right) = \pi_t^e\left(\frac{1}{1+\lambda}\right) = \left[\frac{\lambda}{1+\lambda}(\bar{x} - E_t[\theta_t])\right] = \left[\frac{\lambda}{1+\lambda}\bar{x}\right]$$
$$\Leftrightarrow \pi_t^e = \lambda\bar{x}.$$

Inflation equilibrium is now

$$\pi_t^D = \frac{\lambda}{1+\lambda}(\bar{x} + \bar{x}\lambda - \theta_t) = \frac{\lambda}{1+\lambda}(\bar{x}(1+\lambda) - \theta_t).$$

Output equilibrium is

$$x_t^D = \theta_t + \pi_t - \pi_t^e = \theta_t + \frac{\lambda}{1+\lambda}(\bar{x}(1+\lambda) - \theta_t) - \lambda\bar{x} = \frac{1}{1+\lambda}\theta_t.$$

If $\lambda = 0$, then the government places no weight on output stability and there is no temptation (the incentive fallacy of discretion disappears) to temporarily boost output above its target. Output will then always be equal to θ_t, which was also the case under commitment, and inflation will equal 0, also the case under commitment.

Exam - February 2017

Assignment 2

This is an assignment in price rigidity. The representative agent maximizes

$$U_i = C_i - \frac{1}{\lambda}L_i^\lambda$$

under the constraints of

$$C_i = \frac{P_i}{P}Y_i, \quad Y_i = L_i, \quad Y_i = \left(\frac{P_i}{P}\right)^{-\eta} Y \Leftrightarrow \frac{P_i}{P} = \left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}}.$$

Aggregate output is determined as $Y = M/P$ and standard log-notation holds.

a - Deriving equilibrium output and the aggregate price level Substitution yield the following maximization problem:

$$\max_{Y_i} U_i = Y_i\left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} - \frac{1}{\lambda}Y_i^\lambda \Rightarrow$$

$$\frac{\partial U_i}{\partial Y_i} = 0 \Leftrightarrow -\frac{1}{\eta}Y_i\left(\frac{1}{Y}\right)^{-\frac{1}{\eta}} Y_i^{-\frac{1}{\eta}-1} + \left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} - Y_i^{\lambda-1} = 0 \Leftrightarrow \left(1-\frac{1}{\eta}\right)\left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}} = Y_i^{\lambda-1}.$$

Seeing that $\frac{P_i}{P} = \left(\frac{Y_i}{Y}\right)^{-\frac{1}{\eta}}$ yields

$$\left(1-\frac{1}{\eta}\right)\frac{P_i}{P} = Y_i^{\lambda-1}.$$

Taking logs gives

$$\log(1-\frac{1}{\eta}) + p_i - p = (\lambda-1)y_i \Leftrightarrow y_i = \frac{1}{\lambda-1}(p_i - p) + \frac{1}{\lambda-1}\log(1-\frac{1}{\eta}).$$

Imposing equilibrium (and homogeneity) of $y_i = y$ and $p_i = p$ yields an expression for aggregate output

$$y = \frac{1}{\lambda - 1}\log(1 - \frac{1}{\eta}).$$

Since $y = m - p$, an expression for aggregate price can be derived from the above as

$$m - p = \frac{1}{\lambda - 1}\log(1 - \frac{1}{\eta}) \Leftrightarrow p = m - \frac{1}{\lambda - 1}\log(1 - \frac{1}{\eta}) = m + \mu, \quad \mu = -\frac{1}{\lambda - 1}\log(1 - \frac{1}{\eta}).$$

b - Desired equilibrium price levels for individual goods In order to derive this, ϕ is initially defined as $\phi = \lambda - 1$. Then, we see that

$$\begin{aligned} y_i = \frac{1}{\lambda - 1}(p_i - p) + \frac{1}{\lambda - 1}\log(1 - \frac{1}{\eta}) \Rightarrow m - p + \mu = \frac{1}{\phi}(p_i^\star - p) \\ \Leftrightarrow p_i^\star = \phi m + (1 - \phi)p + \phi\mu = \phi m + (1 - \phi)p + c, \quad c = \phi\mu, \end{aligned}$$

which is the desired expression.

c - Price lock-in Now, if $c = 0$ and prices follow a half-and-half fixed price setting schedule, then

$$p_t = \frac{1}{2}(x_t + x_{t-1}).$$

Expected money supply is white noise with expected value is 0 and certainty equivalence is assumed such that

$$x_t = \frac{1}{2}(p_{i,t}^\star + E_t(p_{i,t+1}^\star)),$$

a difference equation for x_t can be obtained.

$$\begin{aligned} x_t &= \frac{1}{2}(\phi m_t + (1 - \phi)(\frac{1}{2}(x_t + x_{t-1})) + E_t(\phi m_{t+1} + (1 - \phi)\frac{1}{2}(x_{t+1} + x_t))) \\ \Leftrightarrow x_t &= \frac{1}{2}(\phi m_t + (1 - \phi)(\frac{1}{2}(x_t + x_{t-1})) + (1 - \phi)\frac{1}{2}(E_t[x_{t+1}] + x_t)) \\ \Leftrightarrow x_t\left[1 - \frac{1}{2}(1 - \phi)\right] &= x_t\left[\frac{1 + \phi}{2}\right] = \frac{\phi}{2}m_t + \frac{1 - \phi}{4}(E_t[x_{t+1}] + x_{t-1}) \\ \Leftrightarrow x_t &= \frac{\phi}{1 + \phi}m_t + \frac{1 - \phi}{2(1 + \phi)}(E_t[x_{t+1}] + x_{t-1}). \end{aligned}$$

d - Guessing a solution If the guessed solution of $x_t = \beta x_{t-1} + \gamma m_t$ holds, then the expectational term can be eliminated as $E_t[\beta x_t + \gamma m_{t+1}] = \beta x_t$, which gives

$$x_t = \frac{\phi}{1+\phi} m_t + \frac{1-\phi}{2(1+\phi)}(\beta x_t + x_{t-1})$$
$$\Leftrightarrow x_t \left(1 - \frac{\beta(1-\phi)}{2(1+\phi)}\right) = \frac{\phi}{1+\phi} m_t + \frac{1-\phi}{2(1+\phi)}(x_{t-1})$$
$$\Leftrightarrow x_t \left(\frac{2(1+\phi) - \beta(1-\phi)}{2(1+\phi)}\right) = \frac{\phi}{1+\phi} m_t + \frac{1-\phi}{2(1+\phi)}(x_{t-1})$$
$$x_t = \frac{2\phi}{2(1+\phi) - \beta(1-\phi)} m_t + \frac{1-\phi}{2(1+\phi) - \beta(1-\phi)} x_{t-1}.$$

Thus,

$$\beta = \frac{1-\phi}{2(1+\phi) - \beta(1-\phi)} \quad \text{and} \quad \gamma = \frac{2\phi}{2(1+\phi) - \beta(1-\phi)}.$$

Imposing the condition of $\lambda = 1$ means that $\phi = \lambda - 1 = 0$. The equilibrium values for the solution is then

$$\gamma = 0$$
$$\beta = \frac{1}{2-\beta} \Leftrightarrow \beta(2-\beta) = 1 \Leftrightarrow 2\beta - \beta^2 - 1 = 0 \Rightarrow \beta = 1.$$

This is a standard case, where real rigidity is very high and prices are infinitely non-adjustable.

Price rigidity

This considers an economy, where aggregate demand is

$$y = m - p$$

and there are two types of firms. Price-rigid, p^r, and price-flexible, p^f. Aggregate prices are given as

$$p = (1-q)p^f + qp^r.$$

The price settings schedules are

$$p^f = (1-\phi)p + \phi m$$
$$p^r = (1-\phi)E[p] + \phi E[m].$$

The first step is to find p^f in terms of p^r, m, and the parameters of the model. Inserting p in p^f yields the desired results, when rearranging. One must remember to add and subtract ϕp^r to obtain the wished upon result!

$$
\begin{aligned}
p^f &= (1-\phi)((1-q)p^f + qp^r) + \phi m \\
\Leftrightarrow p^f(1-(1-\phi)(1-q)) &= (1-\phi)qp^r + \phi m \\
\Leftrightarrow p^f(\phi + q - \phi q) = p^f(\phi + (1-\phi)q) &= (1-\phi)qp^r + \phi m \\
\Leftrightarrow p^f &= \frac{(1-\phi)qp^r + \phi m}{\phi + (1-\phi)q} \\
&= \frac{(1-\phi)qp^r + \phi m + \phi p^r(\phi + (1-\phi)q) - \phi p^r(\phi + (1-\phi)q)}{\phi + (1-\phi)q} \\
&= \frac{(1-\phi)qp^r + \phi m + \phi p^r\phi + \phi p^r(1-\phi)q - \phi p^r\phi - \phi p^r(1-\phi)q}{\phi + (1-\phi)q} \\
&= \frac{\phi m - \phi p^r + p^r(\phi + (1-\phi)q)}{\phi + (1-\phi)q} \\
\Leftrightarrow p^f &= p^r + \frac{\phi}{\phi + (1-\phi)q}(m - p^r).
\end{aligned}
$$

Inserting this in the general price level yields

$$
p = (1-q)(p^r + \frac{\phi}{\phi + (1-\phi)q}(m - p^r)) + qp^r = p^r + (1-q)\frac{\phi}{\phi + (1-\phi)q}(m - p^r).
$$

Now, p^r is found as a function of the parameters and $E[m]$, using $E[p^r] = p^r$, as

$$
\begin{aligned}
p^r &= (1-\phi)E[p^r + (1-q)\frac{\phi}{\phi + (1-\phi)q}(m - p^r)] + \phi E[m] \\
\Leftrightarrow p^r(1-(1-\phi)) = p^r\phi &= -p^r(1-\phi)(1-q)\frac{\phi}{\phi + (1-\phi)q} + E[m]\left(\phi + (1-\phi)(1-q)\frac{\phi}{\phi + (1-\phi)q}\right) \\
\Leftrightarrow p^r\left(\phi + (1-\phi)(1-q)\frac{\phi}{\phi + (1-\phi)q}\right) &= E[m]\left(\phi + (1-\phi)(1-q)\frac{\phi}{\phi + (1-\phi)q}\right) \\
\Leftrightarrow p^r &= E[m].
\end{aligned}
$$

Now, the general equilibrium price level can be derived as

$$
p = E[m] + \frac{\phi(1-q)}{\phi + (1-\phi)q}(m - E[m]),
$$

while output is given as

$$
\begin{aligned}
y = m - p = m - &\left(E[m] + \frac{\phi(1-q)}{\phi + (1-\phi)q}(m - E[m]) \right) \\
= m &\left(1 - \frac{\phi(1-q)}{\phi + (1-\phi)q} \right) - E[m] \left(1 - \frac{\phi(1-q)}{\phi + (1-\phi)q} \right) \\
&(m - E[m]) \left(\frac{\phi + (1-\phi)q - \phi(1-q)}{\phi + (1-\phi)q} \right) \\
&\Leftrightarrow y = (m - E[m]) \left(\frac{q}{\phi + (1-\phi)q} \right).
\end{aligned}
$$

As $q \to 0$, all firms become price-flexible. For the general price level and output, this means that

$$p = E[m] + \frac{\phi}{\phi}(m - E[m]) = E[m] - E[m] + m = m$$

$$y = 0.$$

Prices are then fully flexible, and the unexpected output deviation of $m - E[m]$ disappears. For $y = 0$, this is interpreted as output never deviating from its expected value.