Main points from Microeconomics II

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These notes cover the main topics from Microeconomics II as taught in spring 2021 by Johannes Wohlfart. The notes are supplemented by relevant chapters in Nechyba 'Intermediate Microeconomics - An intuitive approach with calculus'. Note that the topic of behavioural economics is not included. Equally, taxation and externalities is left out as these topics are treated in previous courses. A few concepts are repeated for ease of the reader in section 9.

1 Principal-agent models and moral hazard

Principal-agent theory covers situations where one part (the principal) is able to make decisions on behalf of another (agent).

Further, moral hazard describes the situation where the one part may increase its exposure to risk as it does not carry the full consequence of doing so.

1.1 Preliminary

We may consider an agent with utility function $u(c)$, where c denotes the options of consumption the agent faces. We will usually assume that the agent has positive marginal utility $u'(c) > 0$ (more is better) and that he is risk averse[1] or risk neutral, $u''(c) \leq 0$.

Note that if $u''(c) < 0$ (strict risk aversion) the requirements for an inner solution are met.

1.2 Contracts with known risk

We consider a case with two possible scenarios H and L which may happen with probabilities p_H or $p_L = 1 - p_H$ (e.g. the probability of hiring an employee of a certain – productive – type). Thus the expected utility is:

$$E(u) = p_H u(c_H, e) + (1 - p_H) u(c_L, e), \tag{1.1}$$

[1] If the second derivative is weakly less than 0 it implies that given some uncertain output c the utility of the expectation of the output $u(E(c)) > E(u(c))$ is larger than the expected value of the utility given that output, the latter being equal to the certainty equivalent. I.e. the agent prefers the situation with no uncertainty (utility of expected value) to the situation with risk (expectation of the utility of the uncertain outcome).

where the individual utility is a function of the outcome c and the effort put in e (which is often a binary function $e \in [0,1]$).

In the case of a insurance contract c_L may consumption options in the case the insurance event happens and c_H may be in the situation it does not. In the case of a work contract it may be whether the worker receives a high or a low wage.

1.2.1 Individual Rationality (IR)

The individual rationality condition means that the agent will only enter the contract if expected utility of the contract $E(u)$ is equal to or larger than his outside option or reservation utility, $\bar{u}$:

$$E(u) = p_H u(c_H,e) + (1 - p_H)u(c_L,e) \geq \bar{u} \tag{IR}$$

In other terms, the expected value of the wage must exceed the outside option. Alternatively, the expected value of the compensation must exceed the expected outcome if no insurance is signed.

When the principal maximises her profits the IR constraint must be binding (unless other constraints are present), as otherwise, the principal may increase her profit (e.g. increase premium or decrease the agent's wage) until the restriction binds.

1.2.2 Dependence on agent's risk aversion

If the agent is strictly risk averse the insurance company (which is risk neutral) will ensure that the consumption options is constant and independent of the insurance event. If the agent is risk neutral he will only accept an actuarial fair insurance.

1.3 Contracts with behaviour dependent risk

Often the agent may affect the outcome through putting in high effort e_H or low (no) effort e_L, either through working harder or through taking preventive actions in the case of insurance.

Given two probabilities p_H and p_L, two constraint for individual rationality arises:

$$p_H u(c_H,e_H) + p_H u(c_L,e_L) \geq \bar{u} \tag{IR_H}$$
$$p_L u(c_H,e_H) + (1 - p_L)u(c_L,e_L) \geq \bar{u} \tag{IR_L}$$

1.3.1 Moral hazard

In case the principal cannot control whether the agent puts in effort or not issues related to moral hazard occurs. In order to solve these issues the principal may incentivise the agent to put in effort, which would require to include (IC) as constraint in the principals problem.

It follows that the agent will put in effort $e \in \{e_L, e_H\}$ if:

$$p_H u(c_H, e_H) + (1 - p_H)u(c_L, e_L) \geq (1 - p_H)u(c_H, e_H) + p_H u(c_L, e_L) \quad (IC)$$

the latter equation being the **incentive compability**.

If the principal may observe or enforce whether the agent puts in effort, there is no longer uncertainty $p_H = 1$. The above then reduces to

$$u(c_H, e_H) \geq u(c_L, e_L) \quad (IC \text{ alt.})$$

In words that is, the utility of the individual given the high consumption options (e.g. wage) and the cost of putting in high effort must exceed the utility from the low consumption options (wage) when the agent only puts in low (no) effort.

Optimal allocation of risk We may show that if one agent is risk averse and another (e.g. an insurance company) is risk neutral it is efficient that the risk neutral agent carries all the risk.

1.4 Information costs and rents

In the case of asymmetric information there might be information costs to one party. For instance the principal might use signaling creates costs for the agents, which has to be compensated.

In cases where the principal has to increase the contract of one agent above its reservation utility in order to prevent that agent of switching to another contract that agent earns informations rents.

2 Monopoly (Ch. 23)

If there's only one seller in a market we have a monopoly. The monopolist faces the market demand D as a function of price p described by:

$$D(p) \quad \text{and} \quad p(x) = D^{-1}(x), \tag{2.1}$$

where $p(x)$ is the inverse demand curve.

Let $TR(x) = p(x) \cdot x$ be the total revenue of the monopolist and let $C(x)$ describe the total cost. Then the problem of the monopolist is to maximise profits:

$$\max_x \Pi = TR(x) - C(x) \tag{2.2}$$

For concave functions of R and strictly convex functions of C there exists an inner solution and the optimal quantity $x^\star$ is given by the first-order condition:

$$MR(x^\star) = MC(x^\star), \tag{2.3}$$

where

$$MR = \frac{\partial TR(x)}{\partial x} = \frac{\partial (p(x)x)}{\partial x} = p(x) + \frac{\partial p(x)}{x}x, \quad \text{and} \quad MC = \frac{\partial C(x)}{\partial x}$$

Noting that $p'(x) < 0$, we see that the marginal revenue from producing an extra unit is the price $p(x)$ plus a penalty $p'(x) \cdot x < 0$ from lowering the price of all the other units x the monopolist has already supplied.

This penalty arises because the monopolist is forced to charge the same price for all units (the monopolist may not conduct *price discrimination*), and gives rise to a dead weight loss. This is opposite to perfect competition, where price p is given (and thus not a function of x). In this case the $MR(x^\star) = p$ and the optimality condition becomes $MC(x^\star) = p$.

2.1 Pitfalls

Notice that if the profit function $\Pi(x) = TR(x) - C(x)$ is not strictly concave, we can have multiple solutions to the first-order condition. We must therefore check the second-order condition and compare actual profits.

The optimal solution may also be a corner solution ($x = 0$). In this case one should divide costs into fixed and variable costs $C(x) = FC + VC(x)$. In the short run fixed costs cannot be avoided and the monopolists problem thus reads

$$\max_x \Pi(x) = TR(x) - \mathbf{VC(x)} \tag{2.2b}$$

2.2 Elasticities

Demand elasticity is – when ξ_h is demanded quantity – given as:

$$\varepsilon_h = -\frac{\frac{d\xi_h}{\xi_h}}{\frac{dp_h}{p_h}}, \tag{2.4}$$

which is a measure of the relative change in demand for a given relative change in price.

2.2.1 Elasiticy in optimum

First we consider marginal revenue in optimum

$$MR(x^\star) = p(x^\star) + \frac{\partial p(x^\star)}{\partial x^\star} x^\star \tag{2.5}$$

We multiply and divide by p:

$$= p(x^\star) + \frac{\frac{\partial p(x^\star)}{\partial (x^\star)} x^\star}{p(x^\star)} p(x^\star) \tag{2.6}$$

$$= p(x^\star) + \frac{\frac{\partial p(x^\star)}{p(x^\star)}}{\frac{\partial (x^\star)}{x^\star}} p(x^\star) \tag{2.7}$$

We realise that the latter term may be rewritten using the definition of elasticity:

$$= p(x^\star) + \frac{1}{\varepsilon(x^\star)} p(x^\star) = p(x^\star)\left(1 + \frac{1}{\varepsilon(x^\star)}\right) \tag{2.8}$$

We further argue that marginal costs are weakly positive, $MC \geq 0$. The first-order condition for the monopolist's problem is $MR = MC$ which implies that marginal revenue must also be positive. Using our rewritten expression we see that:

$$0 \leq MR(x^\star) = p(x^\star)\left(1 + \frac{1}{\varepsilon(x^\star)}\right) \tag{2.9}$$

For $p > 0$ it is then given that

$$-1 \leq \frac{1}{\varepsilon(x^\star)} \tag{2.10}$$

As $\varepsilon < 0$ we don't change the direction of the inequality, i.e.

$$\Leftrightarrow \varepsilon(x^\star) \leq -1 \tag{2.11}$$

This reveals an important result, namely that a monopolist will never choose an internal solution where demand is inelastic $|\varepsilon| \geq 1$, the reason being that he could increase prices with Δ pct., but the demand only decreases with less than Δ pct. and hence $MR = p(x) + p'(x)x$ would increase.

2.2.2 Mark-up

Further we may use our finding from (2.8) to rewrite the optimality condition:

$$MC(x^\star) = MR(x^\star) = p(x^\star)\left(1 + \frac{1}{\varepsilon(x^\star)}\right) \tag{2.12}$$

Isolating p gives us

$$p(x^\star) = \frac{1}{1 + \frac{1}{\varepsilon(x^\star)}} MC(x^\star) = \frac{1}{1 - \frac{1}{|\varepsilon(x^\star)|}} MC(x^\star) \tag{2.13}$$

This result reveals that the monopolist sets a price equal to the marginal cost times a **markup** depending on the price elasticity of demand. The less elastic the demand (low value of ε) the greater the market power. If demand is perfectly elastic $\varepsilon \rightarrow \infty$ then $MC = p$; perfect competition.

2.3 Regulating monopolies

Taxation and subsidies to monopolist We may either tax or subsidise the monopolist in order to obtain the efficient level. We may analyse the problem by either subtracting the level t from the profit Π or adding the subsidy.

Notably, the distribution of surplus depends on how e.g. the subsidy is financed. One could – in theory – subsidise the monopolist and collect a lump-sump tax from the monopolist in order to obtain the efficient level without creating any distortion. Clearly, this requires quite in-depth knowledge of preferences, etc.

Other alternatives Other ways of regulating monopolies include the following

- Subsidising the consumer
- Introduce price controls (equal to marginal cost at efficient quantity)
- The government could take over production of the good
- The monopolist could be allowed to conduct first-degree price discrimination.

3 Price discrimination (Ch. 23)

Price discrimination covers the practice of charging different prices to different individuals for the same product. A requirement for price discrimination is the prevention of sale between agents. It can be separated in the following terms:

First-degree price discrimination Also called *perfect price discrimination.* The monopolist can identify consumer types, prevent resale, and vary prices across consumers as well as for different units sold to one consumer. The firm can capture the consumers' entire surplus as long as resale is prevented.

Mathematically Without the option to price discriminate the monopolist faced a penalty of $p'(x)x$ when selling an extra unit. The penalty arose because she had to decrease the unit-price of all other units. Under perfect price discrimination this is no longer the case. Marginal revenue is given by:

$$MR = p(x) \Leftrightarrow TR(x) = \int_0^x p(t)dt. \tag{3.1}$$

The problem of the monopolist becomes:

$$\max_x TR(x) - C(x) = \int_0^x p(t)dt - C(x), \tag{3.2}$$

with the first-order condition:

$$MR(x^\star) = MC(x^\star) \Leftrightarrow p(x^\star) = MC(x^\star). \tag{3.3}$$

This is the same **efficient** equilibrium condition as under perfect competition – and thus the traded quantity becomes the same as under perfect competition.

Second-degree price discrimination The firm cannot identify the consumer's marginal willingness to pay and thus structure nonlinear price schedules to induce consumers to reveal their type in a separating equilibrium, that is an equilibrium where all types of players play different strategies and thus reveals information about their type.

Second-degree price discrimination may also be regarded as a principal-agent issue with asymmetric information. In order to incentivise one group, the principal (the monopolist) may want to decrease the quality (which we may model by using quantity as a 'pseudo' for quality) of the poorer option, e.g. in the case of flight tickets.

Another example of second degree price discrimination is selling 'packages' of products at different prices (e.g. different menues with different content).

Mathematically Let S_A and S_B be the *total* prices of packages consisting of x_A and x_B units (of quality) sold to consumer A and B. Further, let the consumer surplus of consumer A be given as:

$$CS_A(x) = \int_0^x p_A(t)dt \tag{3.4}$$

which is the maximum consumer A will pay for a deal of x_A units. Define consumer surplus for consumer B symmetrically. The monopolists problem becomes:

$$\max_{S_A, x_A, S_B, x_B} S_A + S_B - C(x_A + x_B) \tag{3.5}$$

$$\text{s.t.} \quad CS_A(x_A) \geq S_A \text{ and } CS_B(x_B) \geq S_B, \tag{IR_A and IR_B}$$

$$CS_A(x_A) - S_A \geq CS_A(x_B) - S_B, \tag{IC_A}$$

$$CS_B(x_B) - S_B \geq CS_B(x_A) - S_A. \tag{IC_B}$$

The individual rationality conditions ensure that the consumer surplus of accepting the package is larger than or equal to the total price. The incentive compability conditions ensure that consumers are incentivised to choose the packages designed for them. In this process the type of consumers with high willingness to pay often earn information rents.

Third-degree price discrimination The monopolist can identify consumers' marginal willingness to pay. They are restricted to charging per-unit prices, but are not restricted to charging the *same* per-unit price to all consumers, and thus charges different prices depending on consumers willingness to pay.

Examples involves situations where the seller can separate consumers (willingness to pay) on observable characteristics, such as youth/pensioner tickets, student discounts, etc.

Mathematically Note that under third-degree price discrimination we face two or more different types of consumers with different demand functions, we therefore solve the issue by aggregating the demand curves (see section 9.1). The problem of the monopolist becomes:

$$\max_{x_A, x_B} p_A(x_A)x_A + p_B(x_B)x_B - C(x_A + x_B) \tag{3.6}$$

Comparison Note that the profit of no price discrimination is less than the profit from second-degree price discriminination which is less than the profit from first-degree price discrimination.

$$\Pi_0 < \Pi_2 < \Pi_1 \tag{3.7}$$

Equally, the profit from no price discrimination is less than the profit of third-degree price discrimination which is less than the profit of first-degree price discrimination.

$$\Pi_0 < \Pi_3 < \Pi_1 \tag{3.8}$$

However we cannot unambiguously compare the profit from second-degree price discrimination to that of third-degree price discrimination, as it depends on the relative size of the different type of consumers under second-degree price discrimination.

4 Game Theory (Ch. 24)

In this treatment we consider only *simultaneous games* with *perfect information*; i.e. **pure strategies**. We will define terms in the case of multiple players – clearly these may be simplified to the two-agent case.

4.1 Simultaneous game

A (simultaneous) game with N players consists of a set a possible strategies for each players

Strategy set A set of possible strategies for each player $S_1, S_2, ..., S_n$.

Utility functions Utility functions for each player's utility as a function of the strategies selected, $u_i(s_i, s_{-i})$, such that

$$u_i \ : \ S_1 \times S_2 \times ... \times S_N \to \mathcal{R}, \tag{4.1}$$

where s_{-i} is a vector that contain all the strategies but that of individual i.

4.2 Best response function

A best response function $s_i^\star(s_{-i})$ indicates an optimal strategy for player i as a function of the other agents' strategies:

$$s_i^\star \ : \ S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times s_N \rightarrow S_i. \tag{4.2}$$

$s_i^\star$ solves the problem

$$\max_{s_i} u_i(s_i,s_{-i}) \tag{4.3}$$

4.3 Nash equilibrium

A Nash equilibrium is a set of strategies for all players

$$(\bar{s}_1, \bar{s}_2,...,\bar{s}_N) \in S_1 \times S_2 \times ... \times S_N \tag{4.4}$$

which are best response functions to each other:

$$s_i^\star(s_{-i}) = \bar{s}_i \quad \text{for all } i \tag{4.5}$$

As a result it follows that in a Nash equilibrium each agent has maximised utility given the strategies of the other individuals.

4.3.1 Finding Nash equilibria – Discrete case

When dealing with discrete strategies we may display the game in a matrix as seen in 1. In order to identify any Nash equilibrium we:

1. Underline the best response for player 1 given player 2 plays her first strategy (A)
2. Underline the best response for player 1 given player 2 plays her second strategy (B)
3. Continue through all strategies of player 2.
4. Repeat the three steps above for player 2 holding the strategies of player 1 fixed.

Any output with two underlinings is a Nash equilibrium. We may check Nash equilibria by examining whether either player has an incentive to deviate from their strategy.

Player 2 / Player 1	Strategy A	Strategy B
Strategy A	(3,4)	$(-3,\underline{6})$
Strategy B	$(\underline{5},-2)$	$(\underline{0},\underline{0})$

Tabel 1: Matrix visual of a simultaneous game

4.3.2 Finding Nash equilibria – Continuous case

In the case that the set of strategies is no longer discrete but continuous, we cannot display the game in a matrix. In order to find a Nash equilibrium, we solve the problem:

$$\max_{s_i} u_i(s_i,s_{-i}) \quad \text{for all } i \tag{4.6}$$

We may do so through finding the first-order conditions:

$$0 = \frac{\partial u_i(s_i,s_{-i})}{\partial s_i} \quad \text{for all } i, \tag{4.7}$$

which return a set of best return functions (that are functions of the other individuals' strategies):

$$s_i^{\star}(s_{-i}) \quad \text{for all } i \tag{4.8}$$

The Nash equilibrium ($\bar{s}_i$) is then given as the situation where all players play best responses to each other. That is, we insert the best response functions and solve for $\bar{s}_1, \bar{s}_2,...,\bar{s}_N$;

$$s_i^{\star}(\bar{s}_{-i}) = \bar{s}_i \quad \text{for all } i. \tag{4.9}$$

4.4 Dominating and dominated strategies

We say that a strategy s_i' is strictly dominating if

$$\begin{aligned} & u_i(s_i',s_{-i}) > u_i(s_i,s_{-i}) \\ & \quad \text{for all } s_{-i} \in S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times s_N \to S_i \end{aligned} \tag{4.10}$$

That is, s_i' makes player i strictly better regardless of what strategies the other individuals play. Conversely we say that s_i is strictly dominat**ed**.

If we exchange $>$ by $\geq$ we say that the strategy is *weakly* dominating.

5 Oligopoly (Ch. 25)

Oligopoly describes a market setting with a finite number of competitors allowing for market power. We consider two types of competition: Price competition (Bertrand) and Quantity competition (Cournot).

5.1 Bertrand competition

Under Bertrand competition firms will set **prices equal to marginal costs**:

$$p^B = MC, \tag{5.1}$$

because each firm will undercut each other until they reach their marginal cost.

This implies that the total output will equal the quantity under full competition. Note that, in the case of linear demand and fixed marginal costs, the monopoly quantity is half of the 'competitive' quantity (MR intersects the primary axis at half the length of the intersection of the demand curve), thus in case there are two firms in the market, *each* firm will produce the same quantity as the monopolist.

Note that in the case of a discrete strategy set (e.g. integer units) it may be that both firms have positive profits. Assume both firms set prices 1 unit above marginal cost. Then both firms will earn positive profits, but will have no incentive to increase prices as the other firm would take all profits in that case. Nor would they undercut each other by setting prices equal to marginal costs as that would also incur 0 profits. Hence, in the discrete case we have a total of two Nash equilibria; one equal arising from setting prices equal to MC from the start and one from setting prices 1 unit above MC.

5.1.1 Different marginal costs

In the case the firms face different marginal costs, the producer with the lowest marginal cost will capture the whole market under Bertrand competition. The firm with the lowest marginal cost will play a best response by setting the price marginally below the marginal cost of the competitor resulting in higher prices and lower quantity (given common assumptions for demand).

5.1.2 Assumptions

Bertrand competitions assumes that the producers may produce an infinite quantity at the given price, which may seem compromising in some settings.

Further we assume that firms interact only once and simultaneously, however repeated interaction could open for the possibility of 'trigger strategies'. In infinitely repeated interactions we might suspect that prices rise above marginal cost.

5.2 Cournot competition

Under Cournot competition firms firms compete on the quantity supplied.

5.2.1 Solving for optimal quantity

In a two-firm setting we find the Nash equilibria in the following way:

1. Find the residual demand given the quantity supplied of the other firm Q_2, i.e. $Q_1^r(\bar{Q}_2)$
2. Find the inverse demand $P(Q_1,Q_2)$, i.e. price as a function of total output.
3. The firm's profit is then:

$$\Pi_1 = TR_1 - TC_1 = p(Q_1,Q_2)Q_1 - MC_1Q_1, \qquad (5.2)$$

where TR_1 is the **total revenue** of firm 1 and TC_1 is total cost which is the product of the marginal cost and the quantity given constant marginal cost.

4. The firm's problem is then given as:

$$\max_{Q_1} \Pi_1 = p(Q_1,Q_2)Q_1 - MC_1 Q_1 \tag{5.3}$$

which is equivalent to solving:

$$MR_1 = MC_1 \tag{5.4}$$

That is find the first order conditions of the profit and solve for the optimal quantity Q_1.

5. Solve the above for the second firm. If the firms are identical the process is symmetrical for firm 2

6. Insert $Q_2(Q_1)$ in $Q_1(Q_2)$ and solve for the Cournot output level Q_1^C which is the best response given your production. If firms are identical $Q_1^C = Q_2^C$.

Note, that there is no weakly dominating strategy under Cournot competition, as the best response *depends* on the response of the other agent.

5.3 Comparison of Cournot and Bertrand

We note that the quantity under Bertrand competition is equal to the quantity under perfect competition $Q^\star$ and is higher than that of Cournot competition, which is again higher than the monopoly quantity:

$$Q^M < Q^C < Q^B = Q^\star \tag{5.5}$$

Equally for prices

$$p^M > p^C > p^B = p^\star, \tag{5.6}$$

where $p^\star$ is the price under perfect competition.

Dead weight loss The dead weight loss of Cournot competition will be less than that of monopoly but larger than under perfect competition. Under Bertrand competition there will be no dead weight loss

Other differences In the case of Bertrand competition, if the firm makes a small error in its price setting, it looses all profits. In the Cournot case it will lose some profits compared to the optimal level, but not all. Thus the Bertrand model fits better to a market with no frictions and completely identical products.

6 Public goods (Ch. 27)

Public goods are characterised by being *non-rival* and *non-excludable*. This implies that the consumption of one agent is equal to the consumption of the other, $g_1 = g_2 = g$.

Further we commonly model public goods in an economy where agents are given an initial endowment e_i which can be spent either consuming a private good x_i (e.g. money) or a public good g which can be purchased using the private good using a cost function $c(g)$.

Graphically, we notice that addition of public goods is *vertical* whereas the addition of private goods is horisontal.

6.1 Decentralised provision of public goods

Due to the nature of under private production there will be an underproduction of public goods (as for the production of private goods with positive externalities).

Assuming that all agents can produce the public good $g_A + g_B = g$, the agent maximises her utility taken the decision of other agents as given. Thus,

$$\max_{x_A, g} u_A(x_A, g) \tag{6.1}$$
$$\text{s.t. } e_A = x_A + c(g_A), \quad g = g_A + g_B.$$

Which may be simplified to

$$\max_{g_A} u_A(e_A - c(g_A), g_A + g_B),$$

where we've inputted the constraint directly into the agent's utility function. The first-order condition [2] is given as:

$$0 = \frac{\partial u_A}{\partial(e_A - c(g_A))}\frac{\partial}{\partial g_A}(e_A - c(g_A)) + \frac{\partial u_A}{\partial(g_A + g_B)}\frac{\partial}{\partial g_A}(g_A + g_B) \tag{6.2}$$
$$= \frac{\partial u_A}{\partial x_A} \cdot -\frac{\partial c(g_A)}{\partial g_A} + \frac{\partial u_A}{\partial g} \cdot 1 \tag{6.3}$$
$$= -c'(g_A)\frac{\partial u_A}{\partial x_A} + \frac{\partial u_A}{\partial g} \tag{6.4}$$

We rewrite the expression as

$$MC(g_A) = \frac{\frac{\partial u_A}{\partial g}}{\frac{\partial u_A}{\partial x_A}} \equiv |MRS_A| \tag{6.5}$$

[2] We seek to find the total derivative of a function on the form:

$$\frac{d}{dx}f(h(x),j(x)) = \frac{\partial f}{\partial h(x)}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial j(x)}\frac{\partial j(x)}{\partial x},$$

where we've implemented the chain rule.

We conduct a symmetrical derivation for agent B, which leads us to conclude that decentralised provision of goods has the solutions:

$$|MRS_A| = MC(g_A) \tag{6.6}$$
$$|MRS_B| = MC(g_B) \tag{6.7}$$

which compares to the efficient condition:

$$|MRS_A| + |MRS_B| = MC(g_A + g_B), \tag{6.8}$$

as derived in the following subsection.

6.2 Socially optimal provision of public goods

The efficient provision of a public good (or public bad) may be found through solving the problem:

$$\begin{aligned} &\max_{x_a, x_b, g} u_A(x_A, g) \\ &\text{s.t. } u_B(x_B, g) \geq \bar{u}, \\ &\quad e_A + e_B = x_A + x_B + c(g). \end{aligned} \tag{6.9}$$

That is maximising the utility of one agent, holding the utility of the other agent(s) constant will result in a (Pareto) efficient solution as we won't be able to increase the utility of one of the agents without making the other worse off.

We may write the Lagrangian as:

$$\begin{aligned} \mathcal{L}(x_A, x_B, g) = {} & u_A(x_A, g) \\ & + \lambda_1(\bar{u} - u_B(x_B, g)) + \lambda_2(e_A + e_B - (x_A + x_B + c(g))) \end{aligned} \tag{6.10}$$

There are a total of five first-order conditions, however for brevity we leave out differentiating with regards to the Lagrangian multipliers – further $\mathcal{L}$ is really a function of five variables, including the Lagrangian multipliers.

The first-order conditions are given as:

$$0 = \frac{\partial \mathcal{L}}{\partial x_A} = \frac{\partial u_A}{\partial x_A} - \lambda_2 \tag{6.11}$$
$$0 = \frac{\partial \mathcal{L}}{\partial x_B} = -\lambda_1 \frac{\partial u_B}{\partial x_B} - \lambda_2 \tag{6.12}$$
$$0 = \frac{\partial \mathcal{L}}{\partial g} = \frac{\partial u_A}{\partial g} - \lambda_1 \frac{\partial u_B}{\partial g} - \lambda_2 \frac{\partial c(g)}{\partial g}, \tag{6.13}$$

which we rewrite as

$$\lambda_2 = \frac{\partial u_A}{\partial x_A} \tag{6.14}$$
$$-\frac{\lambda_2}{\lambda_1} = \frac{\partial u_B}{\partial x_B} \tag{6.15}$$
$$\frac{1}{\lambda_2}\frac{\partial u_A}{\partial g} - \frac{\lambda_1}{\lambda_2}\frac{\partial u_B}{\partial g} = \frac{\partial c(g)}{\partial g} \tag{6.16}$$

Inserting (6.14) and (6.15) in the latter experience yields

$$\frac{\frac{\partial u_A}{\partial g}}{\frac{\partial u_A}{\partial x_A}} - \frac{\frac{\partial u_B}{\partial g}}{\frac{\partial u_B}{\partial x_B}} = \frac{\partial c(g)}{\partial g} \tag{6.17}$$

Finally we notice that this expression equals that sum of the (absolute) marginal rates of substitution:

$$|MRS_A(x_A,g)| + |MRS_B(x_B,g)| = c'(g) = MC_g, \tag{6.18}$$

where MRS is the agent's willingness to give up one unit of the private good x_i for one unit of the public good g. We may increase the level of the public good until the marginal benefits from converting the private good into the public good equals the marginal cost MC_g of producing one extra unit of the public good.

6.2.1 Quasilinear utility

Under quasilinear preferences (on the form $u_A(x_A,g) = v_A(g) + x_A$) we notice that $\frac{\partial u_A}{\partial x_A} = \frac{\partial u_B}{\partial x_B} = 1$ that enter the denominator of MRS. This means the efficiency condition is independent of the distribution of other goods (in this case the private good). This is consistent with our knowledge that the quasilinear tastes are unaffected by income effects.

6.2.2 Comparison to private provision

We note that the production under private provision is smaller than the efficient provision. Assuming decreasing marginal returns to the public goods $\frac{\partial^2 u_A}{(\partial g)^2} < 0$ means that MRS is decreasing[3], in g. This in turn implies that the equilibrium under private provision will have too little public goods.

We notice that given the other agents' provision of the public good, agent A has incentive to *free ride*, i.e. lower her own provision. Further the free-riding problem will increase with the number of agents in the economy.

There is a level of *crowding out* such that if agent A provides another unit, agent B provides less. Under quasilinear tastes there will be full crowding out (one-to-one reduction).

6.3 Lindahl prices

Due to the non-excludable nature of public goods, a firm or public entity may only produce one quantity of a public good g which is consumed by all agents

[3]Using that $\frac{\partial^2 u_A}{(\partial g)^2} < 0$ we see that

$$\frac{\partial MRS}{\partial g} = \frac{\partial}{\partial g}\frac{\partial u_A}{\partial g} \cdot \frac{1}{\frac{\partial u_A}{\partial x_A}} = \frac{\partial^2 u_A}{(\partial g)^2} \cdot \frac{1}{\frac{\partial u_A}{\partial x_A}} < 0,$$

in the economy. Hence it may seek to charge individualised prices such that 1) all agents demand the amount g of the public good and 2) its costs are covered.

Mathematically the problem for agent A becomes:

$$\max_{x_A,g_A} u_A(x_A,g_A) \quad \text{s.t. } e_A = x_A + t_A g_A, \tag{6.19}$$

where we've set the private of the private good x_A to 1 (numeraire good). We realise that the solution is that the marginal rate of substitution (MRS_A) must equal relative prices (which is just the price of the public good as we've set $p_X = 1$).

$$|MRS_A| = t_A \tag{6.20}$$

and symmetrically for agent B

$$|MRS_B| = t_B \tag{6.21}$$

We've argued that the public entity producing the public good must covers its costs, such that $t_A + t_B = c$, which yields:

$$|MRS_A| + |MRS_B| = c, \tag{6.22}$$

that is the Lindahl equilibrium is efficient.

Comparison of equilibria

Walras equilibrium (private goods) Consumers consume different quantities at the same price.

Lindahl equilibrium (public goods) Consumers consume the same quantity at different prices.

Notice however, that agents in the economy have incentives to lie about their preferences in order to pay lower prices for the public good and in practice it is unlikely that Lindahl equilibrium will be obtained.

6.4 Vickrey-Clarke-Groves Mechanism

We argue that agents have an incentive to lie about their preferences. In order to remove the incentive to lie we seek to design a *mechanism* where truth-telling is a best response. A *mechanism* consist of a set of possible messages M which the N individuals in the economy may send, and a function h which translates the messages of the individuals to an outcome A, i.e. $g : M^N \rightarrow A$.

A mechanism designer seek to define a mechanism (M,h) which returns the same outcome from the individuals messages as had the public entity been able to convey the individuals true preferences.

The mechanism The Vickrey-Clarke-Groves mechanism suggest that:

1. Individuals are asked to reveal their (inverse) demand for the public good with each individual i revealing $RD_i(g)$. In terms of the previous terminology the set of possible messages is a set of revealed demand functions. The revealed demand is the solution to maximising the agent's utility $u_i(x_i,g)$, where x_i is the private good and g is the amount of the public good.

2. The public entity providing the public good then sets the level of the public good g by equating the sum of the agent's revealed demands and the marginal cost of producing that good:

$$\sum_i RD_i(g^\star) = MC_g \tag{6.23}$$

3. If the public good is produced, each individual is charged a price k_i that ensures full funding: $\sum_i k_i = MC_g$

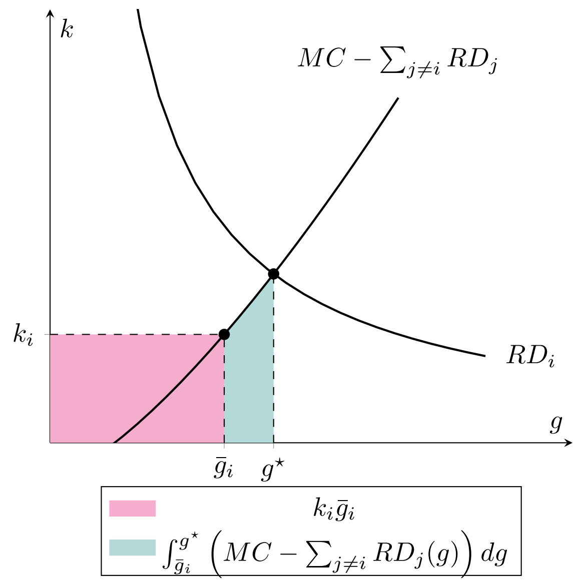
4. In order to prevent incentives to lie we introduce a *Clarke tax* which determines a quantity $\bar{g}_i$ such that the price k_i equals the difference between the marginal cost of producing the public good and the revealed demand of all the *other* individuals in the economy, $k_i = MC_g - \sum_{j\neq i} RD_j(g)$. Note that the individual's own revealed demand does not affect the quantity $\bar{g}_i$. The individual is 'taxed' for the cost his message incurs on the other individuals in the economy.

   Mathematically the total payment K_i individual i must pay is:

$$K_i(k_i) = k_i\bar{g}_i + \int_{\bar{g}_i}^{g^\star} \left(MC - \sum_{j\neq i} RD_j(g) \right) dg. \tag{6.24}$$

In the above we've considered the case of public good which is continously. In lectures, the case of a discrete public good is considered which introduces the notion of *pivotal* agents which determines if the good is produced or not.

Graphically, we may show this as



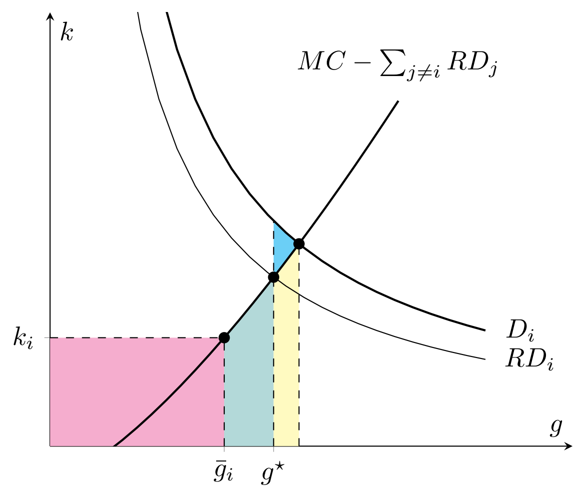
Figur 1: Vickrey-Grove-Clarke Mechanism

6.4.1 Truth telling as dominant strategy

The individual has no control over k_i nor the level $\bar{g}_i$. She may only affect the overall payment she has to pay through affecting $g^\star$.

Assume now that the revealed demand of individual i, RD_i is below her true demand, that is she is underreports as we've illustrated in figure 2. She will save amount equal to the yellow area, but will forego an amount of the public good equal to the yellow and blue areas combined. The net result is that she will be worse off than when reporting her true demand (she will be worse off by the blue area). The analysis for overreporting is symmetrical.

Conclusively, we see that truth telling is a dominant strategy equilibrium under the Vickrey-Clarke-Groves mechanism.



Figur 2: Vickrey-Grove-Clarke Mechanism with underreporting

6.4.2 Feasability of the VCG Mechanism

We notice that we initially chose k_i such that $k_i g^\star$ ensured fully funding. The additional revenue from individual i is thus excess revenue (the green area in figure 1).

Only if the initial prices k_i exactly match such that $\bar{g}_i = g^\star$ is the total costs equal to the total revenue which is the case of *Lindahl* prices.

Alas, what to do with the excess revenue? If we give the revenue back to the agents in the economy income effects would arise. Hence the Vickrey-Clarke-Groves mechanism is inefficient as our only option is to throw away the excess revenue.

Only in the special case of quasilinear preferences – where there are no income effects – is the mechanism efficient; a rather bold assumption!

7 Social preferences (Ch. 28)

This sections introduces new terms regarding social preferences which is relevant for decision making in e.g. governments and politics.

7.1 Social preferences

In a society where agents' preferences differ, we seek to identify a some criteria for how society should make decisions.

Preference relation Let A be a set of possible social outcomes. A preference relation $\succsim$ over A is a quantity of ordered pairs from A indicating which options are preferred over others, i.e. the preference relation is a subset of

A^2; $\succsim \subset A^2$. If $(x,y) \in A^2$ and $(x,y) \in \succsim$ then x is weakly preferred to y and we write $x \succsim y$.

Totality A preference relation $\succsim$ is **total** if for all $(x,y) \in A^2$ either $x \succsim y$ or $y \succsim x$.

In words, totality means we're able to compare all possible pairs.

Transitivity A preference relation $\succsim$ is **transitive** if for all $(x,y,z) \in A^3$ where $x \succsim y$ and $y \succsim z$ it holds that $x \succsim z$.

Note, that although this may seem trivial it does not hold for any preference relation unless it is specified.

Rationality A preference relation $\succsim$ is said to be **rational** if it is *total* and *transitive*.

Utility function Any preference relation defined through a utility function is always rational.

7.2 Decision making

We wish to aggregate the individual preferences to analyse decision making in society.

Social Choice Function (SCF) Consider an economy with N agents. Denote agent i's preference relation $\succsim_i$ and let $\{\succsim\}_A$ denote the set of possible preference relation over a set of possible social outcomes A.

A **social choice function** $f : \{\succsim\}_A^N \to \{\succsim\}_A$ is a function that aggregates the individual preferences into a single social preference ordering $\succsim^\star$, that is:

$$\succsim^\star = f(\succsim_1, \succsim_2, ... \succsim_N) \tag{7.1}$$

In words a social choice function takes the individuals preferences as input and outputs how society ranks certain outcomes A.

Dictatorship SCF In terms of the SCF we may define a dictatorship as

$$f(\succsim_1, \succsim_2, ... \succsim_N) = \succsim_1 \tag{7.2}$$

In words, the overall social preference is only dependent on the preferences of one individual.

Democracy SCF We may define democracy as

$$x \succsim^\star y \text{ if and only if } x \succsim_i y \text{ for a majority of agents} \tag{7.3}$$

Transitivity of SCF It does not follow that rationality of individuals implies rationality of the social choice functions. This explains the emergence of *Condorcet cycles*

Condorcet winner A proposal that defeats all other proposals in pair-wise voting.

Condorcet cycle A situation without a Condorcet winner which means that pair-wise voting may continue with no winner. Arises due to intransitive social preferences.

Agenda setter In the case of no Condorcet winner an agenda setter may determine the winner of pair-wise voting by the order of pair-wise competitions.

Single peaked preferences We say agents have *single peaked preferences* if for each agent i there exists some ideal point $x_i \in A$ for which it holds for all $(z,y) \in A^2$:

$$x_i \geq y \geq z \Rightarrow y \succsim_i z \text{ and } x_i \leq y \leq z \Rightarrow y \succsim_i z \tag{7.4}$$

Median Voter Theorem Assume that agents preferences are single-peaked. We say agent i is a median voter if his ideal point x_i is equal to the median of all the ideal points:

$$\forall j : x_i = \text{med } x_j \tag{7.5}$$

Then the ideal point of the median voter is socially optimal under the democracy SCF:

$$\forall y \in A : \text{med } x_j \succsim^\star y \tag{7.6}$$

In words, if issues fall on a single-dimensioned spectrum and as long as voters have single-peaked preferences, majority rule over pairwise alternatives result in the election of the median voter's ideal point.

7.3 Arrow's Impossibility Theorem

Kenneth Arrow proved that there are no social choice functions that fulfill his five (reasonable) axioms.

7.3.1 Axioms

Arrow argued that a social choice function ought to satisfy the following axioms:

Universal domain (UD) Let $\mathcal{Q}_A \subset \{\succsim\}_A$ be the set of rational preferences over A. The universal domain axiom requires the social choice function to be defined for all rational preferences over A:

$$f : \mathcal{Q}_A \rightarrow \{\succsim\}_A \tag{7.7}$$

Pareto Unanimity (PU) For any pair $(x,y) \in A^2$ it applies that if all agents prefer x over y then the social preferences should do so too:

$$\forall i : x \succsim_i y \Rightarrow x \succsim^\star y, \tag{7.8}$$

where $\succsim^\star = f(\succsim_1, \succsim_2, ... \succsim_N)$.

Rationality (R) The social preferences arising from the social choice function f must be rational.

Independence of Irrelevant Alternatives (IIA) Let $R \in \mathcal{Q}_A^N$ and $R' \in \mathcal{Q}_A^N$ be two set of (rational) preferences for individual agents and let $\succsim^\star = f(R)$ and $\succsim^{\star'} = f(R')$ be the associated social preferences. Then for any pair $(x,y) \in A^2$ it should apply that, if

$$\forall i : x \succsim_i y \Leftrightarrow x \succsim_i' y \tag{7.9}$$

then it should apply

$$x \succsim^\star y \Rightarrow x \succsim^{\star'} y \tag{7.10}$$

In words, if x ranks above y in both sets of individual preferences it shouldn't affect the social preferences. Alas, changing the preference for some alternative z is irrelevant.

Non-Dictatorship For all agents i there must be at least one set of individual preferences $R \in \mathcal{Q}_A^N$ where i does not dictate social preferences:

$$f(R) \neq \succsim_i \tag{7.11}$$

7.3.2 Theorem

Arrow's Impossibility Theorem argues that for $N \geq 2$ and least 3 different social outcomes, then there is *no* social choice function that satisfies the axioms (UD), (PU), (R), (IIA) and (ND).

Alternatively, there exists a social choice function that satisfies (UD), (PU), (R) and (IIA), but it violates the (ND) axiom and therefore results in an Arrow Dictator.

8 Social Utility (Ch. 29)

The following examines the normative aspects of economics, especially the links to microfoundations that imply normative conclusions.

8.1 Utility Possibility Set (UPS)

We consider an Edgeworth economy (see section 9) with two agent each with and endowment e_A and e_B as well as a utility function u_A and u_B both of which are increasing in all arguments. Notably, we assume that the utility is cardinal such that $u_A(0,0) = u_B(0,0) = 0$.

By maximising the utility of agent A given the utility of agent B (and subject to their joint endowments) we are able to find the **utility possibility frontier** which forms the boundary of the utility possibility set.

$$UPS = \{u_A(x_1^A, x_2^A), u_B(x_1^B, x_2^B) | (x_1^A, x_2^A, x_1^B, x_2^B) \in X\}, \tag{8.1}$$

where X is the consumption possibility frontier

$$X = \{(x_1^A, x_2^A, x_1^B, x_2^B) \geq 0 | e_1^A + e_1^B \geq x_1^A + x_1^B \wedge e_2^A + e_2^B \geq x_2^A + e_X^B\} \tag{8.2}$$

8.2 Consequentialism

In the following we take an **consequentialist** approach – which is focused only on outcomes or *consequences* rather than e.g. processes – meaning that it is a *necessary* condition for an optimal outcome to be located on the utility possibility frontier (however it is not a sufficient condition for optimality as we may prefer some outcomes on the frontier over others).

8.2.1 Social Welfare Function (SWF)

We may define a social welfare function $U(u_A, u_B)$. We then seek to maximise social utility as:

$$\max_{u_A, u_B} U(u_A, u_B) \text{ s.t. } (u_A, u_B) \in UPS \tag{8.3}$$

Note that a solution to this problem must also be Pareto efficient as we cannot make someone better of without making another worse of.

Utilitarism The primary case for *consequentialism* is *utilitarism* whose classic proponents where J. Bentham and J.S. Mill. Classic utilitarism envolves on the idea that an action is 'good' (morally right) if and only if it maximises the net utility of all individuals.

Mathematically, we seek to maximise the total (social) utility given what is possible (as defined by the utility possbility frontier X) eventually defining $U(u_A, u_B) = u_A + u_B$. The problem (**??**) thus rewrites as:

$$\max_{x_1^A, x_2^A, x_1^B, x_2^B} u_A(x_1^A, x_2^A) + u_B(x_1^B, x_2^B) \text{ s.t. } (x_1^A, x_2^A, x_1^B, x_2^B) \in X \tag{8.4}$$

Other Social Welfare Function We may consider other functional forms of social welfare functions such as the below:

Harsanyi weights $U(u_A, u_B) = \gamma_A u_A + \gamma_B u_B$

Cobb Douglas $U(u_A, u_B) = u_A^{\beta} u_B^{1-\beta}$

Rawlsian SWF $U(u_A, u_B) = \min(u_A, u_B)$

Equlity and Social Welfare Functions Note that under the utilitarian social welfare function $U(u_A,u_B) = u_A + u_B$ equality plays no role. In the Cobb-Douglas style $U(u_A,u_B) = u_A^\beta u_B^{1-\beta}$ equality plays a role and the most extreme case is that of Rawls which suggest that total utility is equal to the utility of the worst-off.

Rawls emergence of a social welfare function arose from his theory of the 'Veil of Ignorance' implying that an individual ought to decide on the design of society from beneath a veil of ignorance; that is without knowing his own placement in society. Rawl argues that one would choose to regard utilities of different individuals as perfect complements (i.e. resulting in equality).

Eventually Harsanyi[4] argued that we would rather maximise *expected utility* – a point which Rawl refuses possible (from beneath the veil of ignorance).

8.2.2 Issues with Social Welfare Functions

Aggregating utility Reminding ourselves of our assumption of cardinal utility functions and adding the assumption of interpersonal comparison, we may let $U(u_A,u_B)$ be a social welfare function and define a social preference ordering by:

$$(x_1^A, x_2^A, x_1^B, x_2^B) \succsim^\star (x_1^{A'}, x_2^{A'}, x_1^{B'}, x_2^{B'}) \tag{8.5}$$

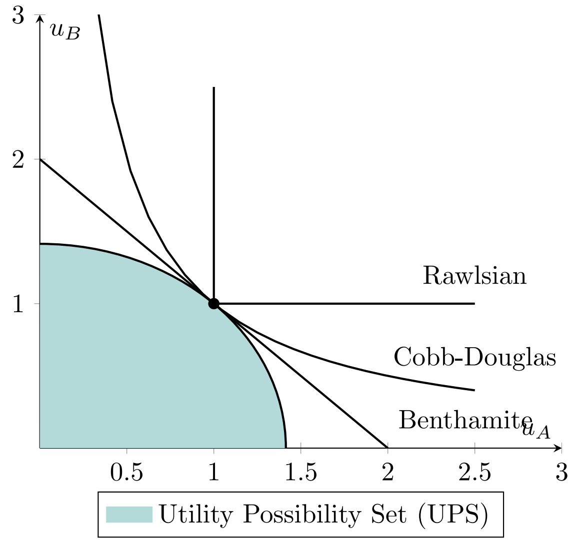
$$\Leftrightarrow U(u_A(x_1^A, x_2^A), u_B(x_1^B, x_2^B)) \geq U(u_A(x_1^{A'}, x_2^{A'}), u_B(x_1^{B'}, x_2^{B'})) \tag{8.6}$$

We may realise that we have now aggregated the individual preferences, but our solution still satisfies Arrow's impossibility theorem

The issue however is that we use cardinal utility rather than ordinal utility. We must now confront the specific level of utility, something which many economists are reluctant to accept; in part due to questions of how should it be measured, how should it be defined, etc.

It further has implications for the discussion of equality as monotone transformations (which previously didn't change the relative utility) may now imply that some transformations value equality whereas others do not.

[4]Note that risk aversion in this case lends itself to a CES-type social welfare function as shown in Nechyba. This function reduces, under certain assumptions, to the above cited.



Figur 3: Utility Possibility Set and Social Welfare Functions

9 Terms and concepts

This sections treats some relevant concepts which have been treated in other courses.

9.1 Horisontal addition of demand curves

We consider a number of demand curves for different types of consumers $D_A(p)$ and $D_B(p)$.

1. Identify at what values of price p the demand is positive $D(p) > 0$. Do so by solving for p when $D(p) = 0$.

2. Find that $D_A(p) > 0$ for $p \in [0,p_1]$ and $D_B(p) > 0$ for $p \in [0,p_2]$.

3. Then the **aggregated demand** is:

$$D(p) = \begin{cases} D_A(p) & \text{if } \; p_1 \leq p < p_2 \\ D_A(p) + D_B(p) & \text{if } \; 0 \leq p < p_1 \end{cases} \tag{9.1}$$

4. When finding the **inverse demand function**, the boundaries are given as:

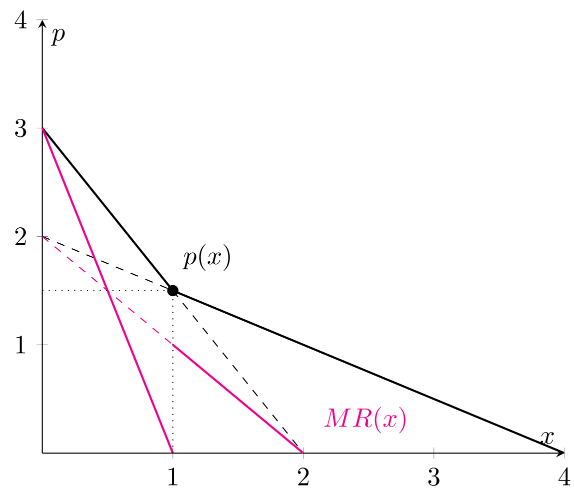
$$p(x) = \begin{cases} D_A^{-1}(p) & \text{if } \; D(p_2) \leq x < D(p_1) \\ (D_A + D_B)^{-1}(p) & \text{if } \; D(p_1) \leq x < D(0) \end{cases} \tag{9.2}$$

Noting that we must take the inverse of the sum of demand functions and not the sum of the inverse demands. We further note that the function is not differentiable at the kink, see figure 4.

5. We define $p_{A+B}(x) \equiv (D_A + D_B)^{-1}(p)$ and $p_A(x) \equiv D_A^{-1}(p)$. Then **marginal revenue** is given as:

$$MR = \frac{\partial}{\partial x} p(x)x = \begin{cases} p_A(x) + p_A'(x)x & \text{if } D(12) \leq x < D(6) \\ p_{A+B}(x) + p_{A+B}'(x)x & \text{if } D(6) \leq x < D(0), \end{cases} \tag{9.3}$$

Note that there is a *jump* in MR around the cutoff value.



Figur 4: Inverse demand of aggregated demand

9.2 Quasilinear preferences

Quasilinear are given on the form:

$$u(x_1,x_2) = v(x_1) + x_2, \tag{9.4}$$

where v is increasing and concave. Quasilinear tastes describes the utility of an essential and inessential good.

Notably, there are no income effects for quasilinear preferences, which means that the compensated (Hicksian demand) and uncompensated (Marshallian demand) are equal. This further implies that the inverse demand indicates the willingnes to pay for each unit, and that we may use the area beneath the inverse demand curve as the consumer surplus.

9.3 Edgeworth Economy

An Edgeworth economy (or exchange economy) is an economy without production where all agents are endowed with an intial endowment. All efficient trades are located on the *contract curve*. Other examples of economies are e.g. Koopmans economies.

9.4 Lump-sum tax

A lump-sum tax is a *non-distortionary* tax the payment of which cannot be avoided through change of behaviour. A theoretically pleasing concept due to its non-distortionary nature, however quite hypothetical.

9.5 Pigou tax

A Pigou tax is a per unit tax meant to restore the optimal quantity by internalising the negative externality a producer might have. However, this requires that the policymakers know the marginal external costs at the socially optimal level.

Further it will only work in the case where production is directly linked to the externality. In the case where a firm can change the level of the externality without affecting the production a Pigou tax will not work.

In this case one might instead tax the negative externality directly (e.g. through a pollution tax).

9.6 Coase theorem

Coase suggested that: *If all property rights are assigned, the end result will be efficient, no matter how the property rights are assigned.*